Analytic representation of quantum systems

HEND ABDELGADER EISSA

Submitted for the Degree
of Doctor of Philosophy

Faculty of Engineering and Informatics
Department of Computing
University of Bradford

2016

Abstract

Finite quantum systems with d-dimension Hilbert space, where position x and momentum p take values in $\mathbb{Z}_d$(the integers modulo d) are studied. An analytic representation of finite quantum systems, using Theta function is considered. The analytic function has exactly d zeros. The d paths of these zeros on the torus describe the time evolution of the systems. The calculation of these paths of zeros, is studied. The concepts of path multiplicity, and path winding number, are introduced. Special cases where two paths join together, are also considered. A periodic system which has the displacement operator to real power t, as time evolution is also studied.

The Bargmann analytic representation for infinite dimension systems, with variables in $\mathbb{R}$, is also studied. Mittag-Leffler function are used as examples of Bargmann function with arbitrary order of growth. The zeros of polynomial approximations of the Mittag-Leffler function are studied.

Dedicated to
who didn't remain with me in life but
lives in my heart...

Acknowledgements

To God

who helps me and has given me the capability to reach this stage

To my supervisor

Professor A. Vourdas who has encouraged, supported and guided me from the initial beginning to the final stages, and understand my circumstances. I would like to thank Dr. C. Lei for the great collaboration and for his overall support. Moreover, I am indebted to Libyan Cultural Affairs for all the financial support.I would also like to thank all members of the Quantum Information Research Group in the school. Also I wish to thank the School of Electrical Engineering and Computer Science at University of Bradford for helping me during my studies.

To my Mom and Dad

Moments will live between words here forever you were with me in those moments it is for you

To my Sisters, Brothers and Nephews

My friends, my hero, my great support,

your encouragement, and belief, have continuously opened the impossible and made it possible. Thank you is too small compared to your support and great contributions. In the darkest moments of despair you held my hand and kept me rising, you took my pain into your hands, and made them small when they were huge, thank you, this is for you.

Declaration

Some parts of the work presented in this thesis have been published in the following articles :

Eissa, H., Evangelides, P., Lei , C and Vourdas . A."Paths of zeros of analytic functions describing finite quantum systems",Physics LettersA ,Volume 380, Issue 4, (2016).

Contents

List of notation

List of notation	
Notation	Definition
$\mathbb{R}$	Real space
$\mathbb{Z}$	Integer space
$\mathbb{Z}_d$	Integers modulo d
$\mathbb{S}$	$[0, 2\pi] \times \mathbb{Z}$
$\mathcal{S}$	$[ML, (M+1)L] \times [NL, (N+1)L]$
$\mathcal{A}$	$[0, 2\pi] \times \mathbb{R}$
ρ	the order of the growth
τ	the type of the growth

List of Figures

Chapter 1

Introduction

In quantum mechanics, the most general formalism of the quantum systems is usually studied in the context of harmonic oscillator where the variables are continuous. In that case the position and momentum take values in $\mathbb{R}$ (Real numbers). The Harmonic oscillator is an important model system in quantum mechanics. Analytic functions have been used in various contexts in quantum mechanics [9],[11],[18]. The most popular analytic representation is the Bargmann representation in the complex plane for the harmonic oscillator [1],[18],[25],[26]. The general theory of growth of an analytic function and the density of their zeros, is applied to the Bargmann function. As example of the Bargmann function we used the Mittag-Leffler function for the arbitrary complex argument Z, and two parameters $\alpha, \beta \in \mathbb{R}$.

In finite quantum systems with d-dimensional Hilbert space, the position and momentum take values in $\mathbb{Z}_d$ (the integers modulo d). Theta function [51] [57],[63],[64] has been used for an analytic representations of finite systems. Theta function is very important since is a Gaussian when working on discretized circle. It has been shown that these analytic functions representing a quantum state had exactly d zeros in a square cell $\mathbb{S}$. As a result, when the d zeros are known the state of the quantum system can be found. In this thesis we consider the time evolution of finite system and the motion of the d zeros on the torus. For the time dependent system, the d zeros follow closed paths on a torus. The Z paths of d zeros define completely the quantum system, the paths of the zeros provide a substitute description for the standard quantum formalism for finite quantum systems. The Hamiltonians in periodic systems have commensurate eigenvalues. In this case, the system is periodic and the paths of the zeros are closed paths. Interesting results show that in some case each path is characterized by the multiplicity M, and by a pair of winding numbers (w_1, w_2), we consider this in more detail and show that after a period the zeros exchange their positions. Some example of the paths of the zeros using different Hamiltonians and different zeros are given. An interesting periodic system in this thesis is one, which has as time evolution operator the displacement operator to real power t. Several numerical example that illustrates these idea. There are deep links between the zeros of the analytic function and the behaviour of quantum system. The analytical relation between the zeros and the various quantum quantities, would be ideal for this study. The ultimate goal is to develop the full quantum formalism in terms of the zeros, and to derive general laws that describe their motion.

1.1 Structure of the thesis

This thesis consists of five chapters. This first chapter gives a brief introduction and the outline of the thesis.

The second chapter provides a review on phase space methods for quantum mechanics. Some operators that play an important role in quantum mechanics are discussed. The Wigner and Weyl functions are introduced. Furthermore, the Bargmann analytic representation in the complex plane and the growth of these functions, are studied. The Mittag-Leffler function, as Bargmann function with fractional order, is discussed. The analytic properties of the Mittag-Leffler function, are studied and the growth of the Mittag-Leffler function, is introduced. Also various examples of the zeros of polynomial approximations of the Mittag-Leffler function, are given. The Mittag-Leffler function as Bargmann function and its zeros, are considered.

In the third chapter, we present the fundamentals of finite quantum systems. Some important tools in the context of finite systems are given. Also Wigner functions and Weyl functions, as well as their properties, are introduced. An analytic representation of finite quantum systems is studied. Finally, the zeros of analytic functions are considered.

Chapter four discusses the paths of zeros of analytic functions. It considers periodic systems, and introduces concepts like the multiplicity, and winding number of the paths of zeros. The novel part in this chapter is the behaviour of the paths of the zeros using different Hamiltonians and different zeros. We conclude in chapter five with a discussion of the results.

Chapter 2

Quantum mechanics on a Real Line

2.1 Introduction

Quantum mechanics [29],[30],[31],[47] studies the behaviour of photons on an atomic scale. The state of a particle is represented by the wave function,which is the solution of the Schrödinger equation. The Hamiltonian eigenvalue problem [or "the time independent Schrödinger equation"] is always in the following form:

$$\mathscr{H}\psi(x) = \mathscr{E}\psi(x). \tag{2.1}$$

Any non-zero solution ψ of this equation is called an energy eigenfunction and the corresponding constant $\mathscr{E}$ is called the energy eigenvalue of the system of Hamiltonian $\mathscr{H}$. $\psi(x)$ is the wave function of the particle in the position representation. The position and momentum take real values, so the phase space is a $\mathbb{R} \times \mathbb{R}$ plane. In the x-representation the momentum and position operators are given by :

$$\begin{aligned} \mathbf{X} &= x \\ \mathbf{P} &= -i\hbar\frac{\partial}{\partial x} \end{aligned} \tag{2.2}$$

In quantum mechanics the variables $\mathbf{X}$ and $\mathbf{P}$ are regarded as operators . The following commutation relation is defined by:

$$[\mathbf{X}, \mathbf{P}] = \mathbf{XP} - \mathbf{PX} = i\hbar \tag{2.3}$$

We are going to set Planck's constant $\hbar = 1$ for simplicity. In the next section, we will introduce the quantum harmonic oscillator , for the one dimensional case. We will also present special states such as number states, displacement and parity operators, coherent states, and the Wigner and Weyl functions. In section 2.4, we give a brief introduction to the Bargmann analytic representation in the complex plane, which is defined by the Glauber coherent states.The Mittag-Leffler function as Bargmann function are discussed in section 2.5. The growth of the Mittag-Leffler function fractional order are introduced in this section.

2.2 Position and momentum

In this section, we investigate the position operator $\mathbf{X}$ and the momentum operator $\mathbf{P}$ in more detail. These operators satisfy the Dirac quantum condition in Eq$(2,3)$.

The position and momentum operators are Hermitian, hence their eigenvalues are real

$$\begin{aligned}\mathbf{X}|x\rangle &= x|x\rangle,\\ \mathbf{P}|p\rangle &= p|p\rangle.\end{aligned} \tag{2.4}$$

Here $|x\rangle$and $|p\rangle$denotes position and momentum states, respectively. The position and momentum operators form orthogonal (not normalised) bases for the Hilbert space $\mathbf{L}^2(\mathbb{R})$ and they satisfy the relation :

$$\begin{aligned}\langle x|x'\rangle &= \delta(x-x'),\\ \langle p|p'\rangle &= \delta(p-p'),\end{aligned} \tag{2.5}$$

where $\delta(x)$ is the Dirac delta function. The eigenstates of the position operator $\mathbf{X}$ and the momentum operator $\mathbf{P}$ satisfy the completeness relation :

$$\begin{aligned}\int_{-\infty}^{\infty} |x\rangle\langle x|dx = 1\\ \int_{-\infty}^{\infty} |p\rangle\langle p|dp = 1\end{aligned} \tag{2.6}$$

We can expand arbitrary state $|f\rangle$ and $|g\rangle$ in terms of the position and momentum basis states as:

$$\begin{aligned} |f\rangle &= \int_{-\infty}^{\infty} dx|x\rangle\langle x|f\rangle \\ |g\rangle &= \int_{-\infty}^{\infty} dp|p\rangle\langle p|g\rangle. \end{aligned} \tag{2.7}$$

Assuming that $\langle x|f\rangle = f(x)$ and $\langle p|g\rangle = g(p)$, then.

$$\begin{aligned} |f\rangle &= \int_{-\infty}^{\infty} dx|x\rangle f(x) \\ |g\rangle &= \int_{-\infty}^{\infty} dp|p\rangle g(p). \end{aligned} \tag{2.8}$$

In this case we call the two wave functions $f(x)$ and $g(p)$ the position representation and the momentum representation of the state $|f\rangle$, respectively. They satisfy the relation :

$$\begin{aligned} &\int_{-\infty}^{\infty} dx|f(x)|^2 = 1 \\ &\int_{-\infty}^{\infty} dp|g(p)|^2 = 1. \end{aligned} \tag{2.9}$$

Where $|f(x)|^2$ and $|g(p)|^2$ are the position and momentum probability density, respectively. The state $|x\rangle$and $|p\rangle$ are related to each other by Fourier Transform :

$$|x\rangle = (2\pi)^{\frac{-1}{2}} \int_{-\infty}^{\infty} dp \exp(-ixp)|p\rangle \tag{2.10}$$

$$|p\rangle = (2\pi)^{\frac{-1}{2}} \int_{-\infty}^{\infty} dx \exp(ixp)|x\rangle. \tag{2.11}$$

The inner product between $|x\rangle$and $|p\rangle$ are:

$$\langle x|p\rangle = (2\pi)^{\frac{-1}{2}} \exp(ixp). \tag{2.12}$$

Here we can write the Fourier transform in terms of Fourier operator, which is defined as :

$$\mathcal{F} = \int_{-\infty}^{\infty} d\xi |\xi\rangle_{x,p}\langle\xi| \tag{2.13}$$

Then :

$$\begin{aligned} \mathcal{F}|\xi\rangle_x = |\xi\rangle_p,\ \mathcal{F}|\xi\rangle_p = |-\xi\rangle_x \\ \mathcal{F}^\dagger X \mathcal{F} = P,\ \mathcal{F}^\dagger P \mathcal{F} = -X, \end{aligned} \tag{2.14}$$

so it is easy to see that :

$$\mathcal{F}^4 = 1. \tag{2.15}$$

The wave function of position states can be expressed in terms of the wave function of momentum states using Fourier transform. These equations are given by the following :

$$f(x) = (2\pi)^{\frac{-1}{2}} \int_{-\infty}^{\infty} dp \exp(-ixp) f(p) \tag{2.16}$$

$$g(p) = (2\pi)^{\frac{-1}{2}} \int_{-\infty}^{\infty} dx \exp(ixp) g(x). \tag{2.17}$$

2.3 Quantum harmonic oscillator

The harmonic oscillator is an extremely important physics problem. In this section we will describe the special case of a quantum system on a real line, which is the one-dimensional linear harmonic oscillator. The potential energy of the system in position x is given by :

$$\mathcal{V} = \frac{1}{2}\mathcal{K}x^2 \tag{2.18}$$

where $\mathcal{K} = m\omega^2$ is the spring constant. For simplicity, let $\mathcal{K} = 1$ and assume that the mass $m = 1$ and angular frequency $\omega = 1$. Then its classical energy is :

$$\frac{p^2}{2m} + \frac{1}{2}\mathcal{K}x^2 = \mathscr{E}. \tag{2.19}$$

Thus, we can set up the Schrdinger equation using the prescription Eq $(2, 1)$. The result is :

$$[-\frac{h^2}{2m}\frac{\mathfrak{d}^2}{\mathfrak{d}x^2} + \frac{1}{2}\mathcal{K}x^2]\psi(x) = \psi(x)\mathscr{E}, \tag{2.20}$$

where $h = m = \mathcal{K} = 1$. In this case, the Hamiltonian operator is given by :

$$\begin{aligned} \mathscr{H} &= -\frac{h^2}{2m}\frac{\mathfrak{d}}{\mathfrak{d}x^2} + \frac{1}{2}\mathcal{K}x^2 \\ \mathscr{H} &= \frac{1}{2}P^2 + \frac{1}{2}X^2. \end{aligned} \tag{2.21}$$

The solution of Eq$(2, 20)$ is :

$$\psi_n(x) = (\frac{2^n\sqrt{\pi}}{n!})\mathbf{H}_n(x)\exp(-\frac{1}{2}x^2). \tag{2.22}$$

Where $\mathbf{H}_n$ is a family of polynomials called the Hermite polynomials and defined as :

$$\mathbf{H}_n(x) = (-1)^n \exp(x^2)\frac{\partial^n}{\partial x^n}\exp(-x^2). \tag{2.23}$$

Instead of working with the operators X and P that satisfy the commutation relation defined in Eq(2.3) and using Eq(2.21) we can define the annihilation and creation operators as :

$$\begin{aligned} \hat{a} &= (2)^{-1/2}(X + iP) \\ \hat{a}^\dagger &= (2)^{-1/2}(X - iP). \end{aligned} \tag{2.24}$$

These two operators obey the canonical commutation relation. For now we note that the position and the momentum operators are expressed by $\hat{a}$ and $\hat{a}^\dagger$ like :

$$\begin{aligned} X &= \frac{(\hat{a} + \hat{a}^\dagger)}{\sqrt{2}} \\ P &= -i\frac{(\hat{a} - \hat{a}^\dagger)}{\sqrt{2}}. \end{aligned} \tag{2.25}$$

Next we calculate the commutator of the creation and annihilation operators.

It is quite obvious that they commute with themselves.

$$[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0. \tag{2.26}$$

Since we know that $[X, P] = -i$ we easily get

$$[\hat{a}, \hat{a}^\dagger] = I. \tag{2.27}$$

The vacuum state $|0\rangle$ is defined using the annihilation operator in Eq(2,24)

$$\hat{a}|0\rangle = 0 \ ; \ \langle 0|0\rangle = 1 \tag{2.28}$$

For the normalised eigenkets $|n\rangle$, the operators $\hat{a}^\dagger$ and $\hat{a}$; act on the number state as follows:

$$\begin{aligned} \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle, \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle. \end{aligned} \tag{2.29}$$

From the relations above, we can also define a number operator $\mathcal{N}$, which has the following property :

$$\begin{aligned} \mathcal{N} &= \hat{a}^\dagger\hat{a} \\ \mathcal{N}|n\rangle &= n|n\rangle. \end{aligned} \tag{2.30}$$

And the Hamilton operator in Eq(2.21) can be expressed as

$$\begin{aligned}\mathcal{H} = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 &= \frac{1}{4}(-(\hat{a} - \hat{a}^\dagger)^2 + (\hat{a} + \hat{a}^\dagger)^2) \\ &= \frac{1}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) = (\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \mathcal{N} + \frac{1}{2}.\end{aligned} \tag{2.31}$$

The scalar product of two number states $|n\rangle, |m\rangle$ is equal to δ_{nm}, which is also equal to the Kronecker delta function :

$$\begin{aligned}\langle n|m\rangle &= \delta_{nm} \\ \sum_{n=0}^{\infty} |n\rangle\langle n| &= 1.\end{aligned} \tag{2.32}$$

2.3.1 Displacement operators and parity operators

The displacement operators $D(\alpha)$ of the harmonic oscillator can be defined as:

$$D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}). \tag{2.33}$$

Where α is a complex number and $\hat{a}^\dagger$ and $\hat{a}$ are the creation - and annihilation - operators.

Properties of the displacement operator

$$D(\alpha)D^{\dagger}(\alpha) = D^{\dagger}(\alpha)D(\alpha) = I$$
$$D^{\dagger}(\alpha) = D(\alpha)^{*} = D(-\alpha) = D(\alpha)^{-1}$$
$$D^{\dagger}(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha$$
$$D(\alpha+\beta) = D(\alpha)D(\beta)\exp(-\frac{1}{2}[\alpha\beta^{*} - \alpha^{*}\beta]). \tag{2.34}$$

We can prove the third Properties as follows:

$$D^{\dagger}(\alpha)\hat{a}D(\alpha) = \hat{a} + [\alpha^{*}\hat{a} - \alpha\hat{a}^{\dagger}, \hat{a}] + \frac{1}{2!}[\alpha^{*}\hat{a} - \alpha\hat{a}^{\dagger}, [\alpha^{*}\hat{a} - \alpha\hat{a}^{\dagger}, \hat{a}]] + ... \tag{2.35}$$

Using Jacobi identity we can note that , the higher order commutators equal to zero then we have :

$$D^{\dagger}(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha. \tag{2.36}$$

We also define the parity operator around the origin as :

$$\mathcal{U}_0 = \int_{-\infty}^{\infty} dx|-x\rangle\langle x| = \int_{-\infty}^{\infty} dp|-p\rangle\langle p|. \tag{2.37}$$

Where this parity operator acting on position or momentum states that :

$$\mathcal{U}_0|x\rangle = |-x\rangle; \qquad \mathcal{U}_0|p\rangle = |-p\rangle. \tag{2.38}$$

And also changes the parameters of the displacement operator as follows:

$$\mathcal{U}_0 D(x_0, p_0)\mathcal{U}_0^\dagger = D(-x_0, -p_0). \tag{2.39}$$

2.3.2 Coherent states

Now we introduce another set of states, the coherent states which play an important role in quantum optics, especially in laser physics([11]- [17]). We will define coherent states as follows : **A coherent state** $|z\rangle$, also called Glauber state, is defined as the eigenstate of the amplitude operator, the annihilation operator $\hat{a}$, with eigenvalues:

$$|z\rangle = D(z)|0\rangle = \exp(z\hat{a}^\dagger - z^*\hat{a})|0\rangle. \tag{2.40}$$

Using the Baker-Campbell-Hausdorff formula one finds:

$$\begin{aligned}\exp(z\hat{a}^\dagger - z^*\hat{a}) &= e^{z\hat{a}^\dagger}e^{-z^*\hat{a}}e^{\frac{1}{2}|z|^2}\\ |z\rangle &= e^{z\hat{a}^\dagger}e^{-z^*\hat{a}}e^{\frac{1}{2}|z|^2}|0\rangle.\end{aligned} \tag{2.41}$$

And since $z|0\rangle = 0$ one has :

$$\begin{aligned}|z\rangle &= \exp(\frac{1}{2}|z|^2)\exp(z\hat{a}^\dagger)|0\rangle\\ &= \pi^{-1/2}\exp(\frac{1}{2}|z|^2)\sum_{n=0}^{\infty}\frac{a^n}{(n!)^{1/2}}|n\rangle.\end{aligned} \tag{2.42}$$

Where z is a complex number.Since $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ we can find that :

$$\hat{a}|z\rangle = \pi^{-1/2}\exp(\frac{1}{2}|z|^2)\sum_{n=0}^{\infty}\frac{z^n}{((n-1)!)^{1/2}}|n-1\rangle = z|z\rangle. \tag{2.43}$$

Using the definition of coherent state we found the scalar product of two coherent states :

$$\langle z|z_1\rangle = \pi^{-1}\exp(z^* z_1 - \frac{1}{2}|z|^2 - \frac{1}{2}|z_1|^2). \tag{2.44}$$

The scalar product of the momentum representation of the coherent state is :

$$\langle p|z\rangle = \pi^{-1/4}\exp(-\frac{p^2}{2} - i\sqrt{2}zp + izz_I), \tag{2.45}$$

where $z = z_R + iz_I$.

The scalar product of position representation and coherent states is equal to:

$$\langle x|z\rangle = \pi^{-1/4}\exp(-\frac{x^2}{2} - \sqrt{2}zx + izz_R) \tag{2.46}$$

The completeness relation for the coherent states is :

$$\frac{1}{\pi}\int_{\mathbb{C}} d^2z|z\rangle\langle z| = 1, \tag{2.47}$$

where $\mathbb{C}$ is the complex plane.

2.3.3 Wigner and Weyl function

One of the common phase formulations of quantum mechanics is based on the Wigner function. It reveals some important properties of the quantum state as described by a density operator . The Wigner function and other phase space distributions describe more clearly the similarities and differences between classical and quantum mechanics. The Wigner function, which was described by E.Wigner in 1932[28],[29],[30],[31],[32],[33], depends on two variables, the position and the momentum. It is defined as follows :

$$\begin{aligned} W(\rho, x, p) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dX \exp(-ipX)\langle x - \frac{X}{2}|\rho|x + \frac{X}{2}\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dP \exp(-iPx)\langle p - \frac{P}{2}|\rho|p + \frac{P}{2}\rangle \end{aligned} \tag{2.48}$$

Where X and P are the position and momentum increments and ρ is the density matrix of the system.

Properties: The Wigner function is not a real probability distribution, and it has the properties :

$$\begin{aligned} \int_{-\infty}^{\infty} dxW(\rho, x, p) = |s(p)|^2; \qquad s(p) = \langle p|s\rangle \\ \int_{-\infty}^{\infty} dpW(\rho, x, p) = |s(x)|^2; \qquad s(x) = \langle x|s\rangle \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdpW(\rho, x, p) = 1 \end{aligned} \tag{2.49}$$

We will prove the first property for $\rho = |s\rangle\langle s|$ where, s is set as follows, using Eq (2.48) to take the integration in bout sides with respect to x:

$$\int_{-\infty}^{\infty} dx W(x,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \exp(-ipX)\langle x - \frac{X}{2}|s\rangle\langle s|x + \frac{X}{2}\rangle dX \quad (2.50)$$

Changes the Variables x and X to $x'$and x'' put $x' = x - \frac{X}{2}$ and $x'' = x + \frac{X}{2}$:

$$\begin{aligned}\int_{-\infty}^{\infty} dx W(x,p) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} \exp(-ip(x''-x'))s(x'')s^*(x')dx'' \\ &= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} s(x'')s^*(x')\frac{1}{\sqrt{2\pi}}\exp(ipx')\frac{1}{\sqrt{2\pi}}\exp(-ipx'')dx'', \quad (2.51)\end{aligned}$$

note that :

$$\begin{aligned}p^*(x') = \langle x'|p\rangle &= \frac{1}{\sqrt{2\pi}}\exp(ipx') \\ p(x'') = \langle x''|p\rangle &= \frac{1}{\sqrt{2\pi}}\exp(-ipx''), \quad (2.52)\end{aligned}$$

using Eq (2.52) in Eq (2.51) we get:

$$\begin{aligned}\int_{-\infty}^{\infty} dx W(x,p) &= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} s(x'')s^*(x')p(x'')p^*(x')dx'' \\ &= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} \langle x''|s\rangle\langle s|x'\rangle\langle p|x''\rangle\langle x'|p\rangle dx'' \\ &= \int_{-\infty}^{\infty} \langle s|x'\rangle\langle x'|p\rangle dx' \int_{-\infty}^{\infty} \langle p|x''\rangle\langle x''|s\rangle dx'', \quad (2.53)\end{aligned}$$

we can insert the completing relation for eigenstates x, p of set s :

$$\int_{-\infty}^{\infty} |x\rangle' \langle x'| dx' = 1$$
$$\int_{-\infty}^{\infty} |x\rangle'' \langle x''| dx'' = 1, \tag{2.54}$$

from Eq (2.54) in (2.53) for $\rho = |s\rangle\langle s|$ we get:

$$\int_{-\infty}^{\infty} dx W(x,p) = \langle p|s\rangle\langle s|p\rangle = |\rho(p)|^2 \tag{2.55}$$

• The Weyl function is given in terms of the displacement operator :

$$\tilde{W}(\rho, X, P) = Tr[\rho D(z)]; \qquad z = X + iP. \tag{2.56}$$

Furthermore, the Weyl functions can be defined by a density operator as follows:

$$\begin{aligned} \tilde{W}(\rho, X, P) &= \int_{-\infty}^{\infty} dX \exp(iPx) \langle x - \frac{X}{2}|\rho|x + \frac{X}{2}\rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dP \exp(-ipX) \langle p - \frac{P}{2}|\rho|p + \frac{P}{2}\rangle. \end{aligned} \tag{2.57}$$

The Wigner and Weyl functions are derived using the two dimensional Fourier transform, as follows :

$$W(x,p) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dX dP \tilde{W}(X,P) \ \exp(-i(Xp - Px)). \tag{2.58}$$

• The Wigner representation of the number states $|n\rangle$ is given as:

$$W(n,x,p) = \frac{(-1)^n}{\pi}\exp(-x^2 - p^2)\mathcal{L}_n[2(x^2+p^2)]. \tag{2.59}$$

And the corresponding Weyl function is :

$$\tilde{W}(n,x,p) = \exp[-\frac{x^2+p^2}{4}]\mathcal{L}_n[\frac{x^2+p^2}{2}] \tag{2.60}$$

Where $\mathcal{L}_n$ are the Laguerre polynomials.

• The Wigner representation of the coherent states $|z\rangle$ is :

$$W(z,x,p) = \frac{1}{\pi}\exp[-(x-\sqrt{2}z_R)^2 - (p-\sqrt{2}z_I)^2]. \tag{2.61}$$

The corresponding Weyl function is given in terms of :

$$\tilde{W}(z,x,p) = \exp[-(\frac{x}{2}+\sqrt{2}iz_I)^2 - (\frac{p}{2}-\sqrt{2}iz_R)^2 - 2|z|^2]. \tag{2.62}$$

In Figure(2.1) we show the Wigner representation of the vacuum state $|0\rangle$. Whereas in Figure(2.2),we show the Wigner representation of the number states (Fock stste) $|1\rangle$, and Figure (2.3) shows the Wigner function of super position of two coherent states $\exp(-(x-3)^2)+\exp(-(x+3)^2)$,:

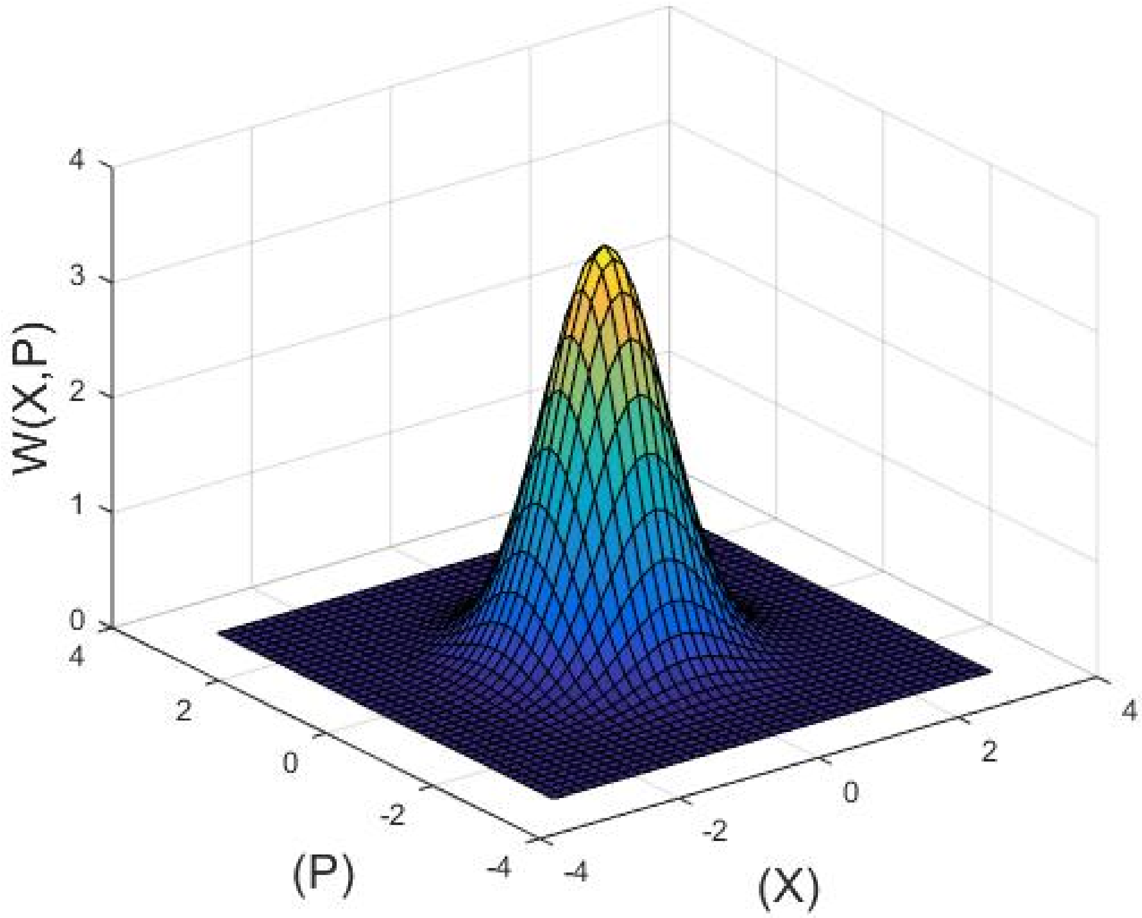

Figure 2.1: Wigner function of the vacuum state $|0\rangle$.

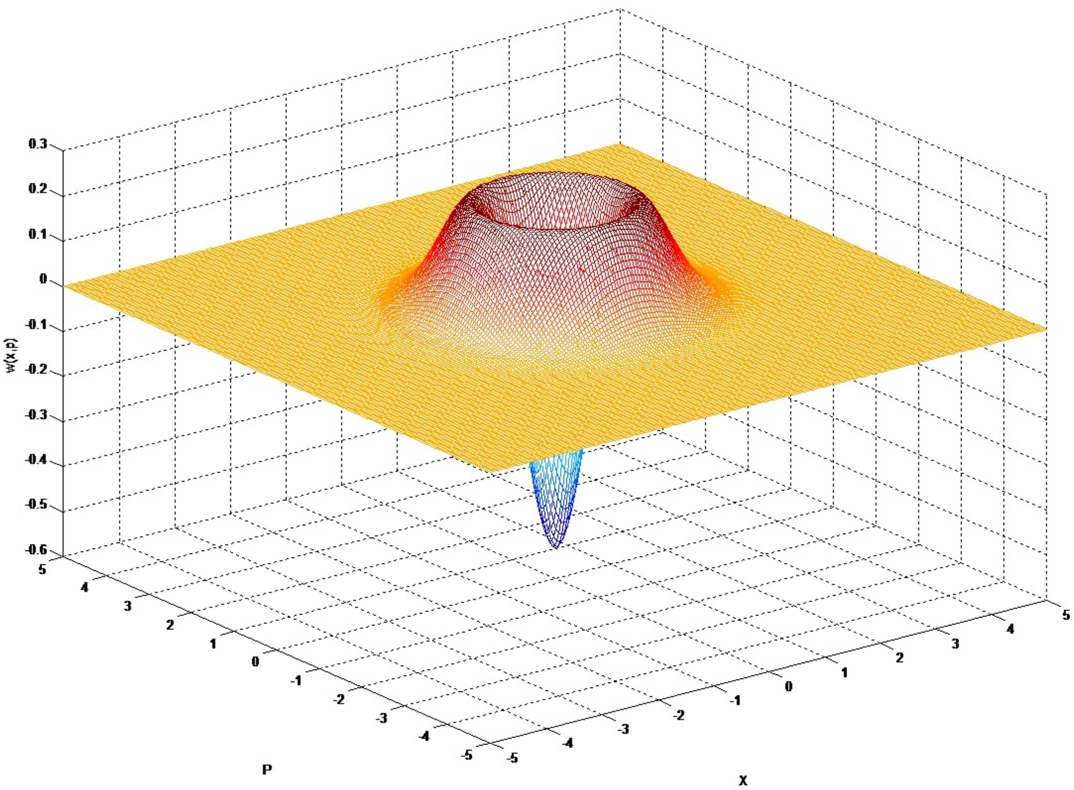

Figure 2.2: Wigner function of the number states (Fock state)$|1\rangle$.

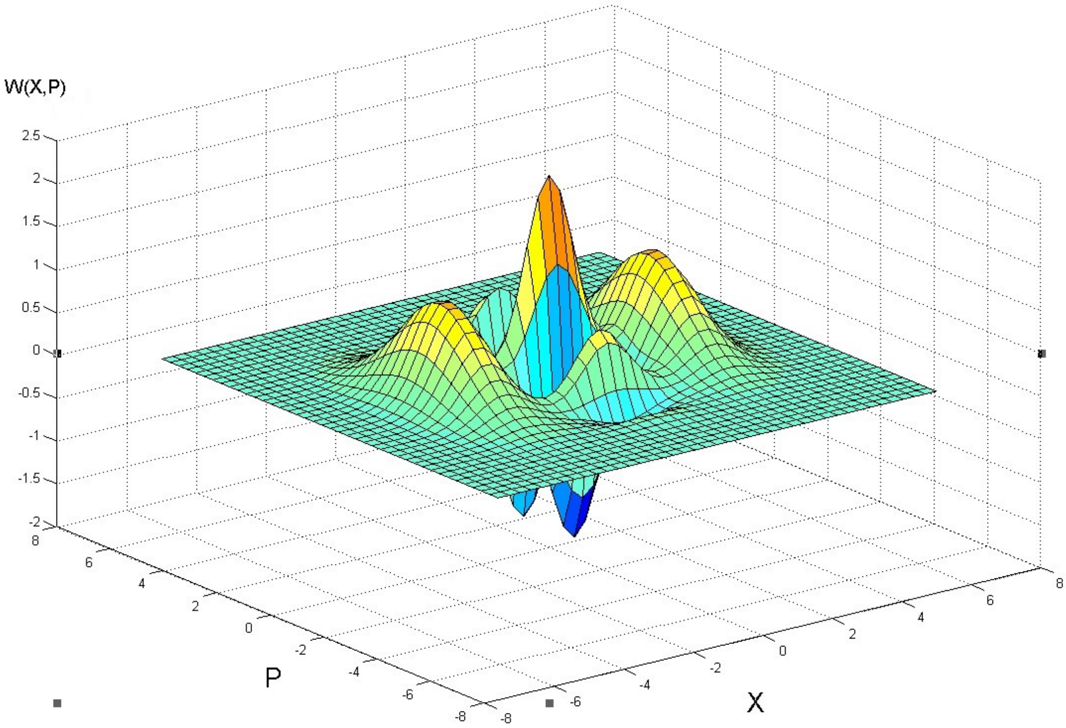

Figure 2.3: Wigner function of super position of two coherent states $\exp(-(x-3)^2)+\exp(-(x+3)^2)$.

2.4 The Bargmann analytic representation

There are several representation that use analytic function. The Bargmann representation is the most well-known one. In this section, we introduce the Bargmann analytic representation in the complex plane defined by the Glauber coherent state. The space of these functions is defined as the space of the entire functions with no singularities. The growth of an analytic function is characterised by its order and type [19],[52],[53],[81],[82],[84].

Let $|g\rangle$ be an arbitrary state :

$$|g\rangle = \sum_{n=0}^{\infty} g_n |n\rangle. \tag{2.63}$$

The normalization condition is given below

$$\sum_{n=0}^{\infty} |g_n|^2 = 1. \tag{2.64}$$

The conjugate of $|g\rangle$ is $\langle g|$ can be written as follows:

$$\langle g| = \sum_{n=0}^{\infty} g_n^* \langle n| \ ; \ |g\rangle^* = \sum_{n=0}^{\infty} g_n^* |n\rangle. \tag{2.65}$$

The Bargmann representation [25],[26],[59], for the state $|g\rangle$ is represented by :

$$g(z) = exp(\frac{|z|^2}{2})\langle z^*|g\rangle = \sum_{n=0}^{\infty} \frac{g_n z^n}{\sqrt{n!}}. \tag{2.66}$$

The inner product of the two states are defined as :

$$\langle g_1|g_2\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2z[g_1(z)]^* g_2(z)\exp(-|z|^2). \tag{2.67}$$

Which can be proved by using the resolution of identity of the coherent states. The Bargmann analytic function can be expressed in terms of the wave function of position and the momentum states as follows:

$$\begin{aligned}\int_{\mathbb{C}} dz_I g(z)\ \exp(-\frac{z_I^2}{2}) = \pi^{3/4}\sqrt{2}\ \exp(\frac{z_R^2}{4})g(-\frac{z_R}{\sqrt{2}})\\ \int_{\mathbb{C}} dz_R g(z)\ \exp(-\frac{z_R^2}{2}) = \pi^{3/4}\sqrt{2}\ \exp(\frac{z_I^2}{4})g(-\frac{z_I}{\sqrt{2}}).\end{aligned} \tag{2.68}$$

The creation and annihilation operator of the Bargmann analytic representation is :

$$\hat{a} = \mathfrak{d}_z, \qquad \hat{a}^\dagger = z. \tag{2.69}$$

•The Bargmann representation for the number states $|n\rangle$ is:

$$g(z) = \frac{z^n}{\sqrt{n!}} \tag{2.70}$$

• The Bargmann representation for the coherent state $|A\rangle$ is:

$$g(z) = exp(Az - \frac{|A|^2}{2}) \tag{2.71}$$

2.4.1 The growth of Bargmann analytic functions and their density of zeros

The growth of an analytic function $g(z)$ is described with two non-negative numbers, (ρ, τ). The order ρ and the type τ. If $M(R)$ is the maximum modulus of $g(Z)$ for $|Z| = R$, and we can write it as such:

$$M(R) = \max_{R=|z|}(|g(z)|) \tag{2.72}$$

Then :

$$\begin{aligned} \rho &= \lim_{R\to\infty} \sup \frac{\ln \ln M(R)}{\ln R} \\ \tau &= \lim_{R\to\infty} \sup \frac{\ln M(R)}{R^{\rho}} \end{aligned} \tag{2.73}$$

In this case $M \approx exp(\tau R^{\rho})$ as $R \to \infty$.

We shall denote as $\mathfrak{R}(\rho, \tau)$ is the set of analytic function in the complex plane with (ρ, τ) smaller than (ρ', τ'), if $\rho < \rho'$ or if both $\rho = \rho'$ and also $\tau < \tau'$.

Convergence of the inner product of Eq(2.67) shows that the maximum growth of the Bargmann function is $(\rho = 2, \tau = 1/2)$, (i.e the Bargmann space is $\mathfrak{R}(2, 1/2)$) .

We next consider a general analytic function $g(z)$,$g(0) \neq 0$, with an infinite number of zeros $\zeta_1, \zeta_2, ..., \zeta_n, ...$, which we label them as :

$$0 < |\zeta_1| \leq |\zeta_2| \leq .... \leq |\zeta_n| \leq ...... \tag{2.74}$$

Multiples zeroes are always assumed to be repeated according to the sequence (ζ_n), (as n goes to infinity). The convergence exponent ϱ and δ of this sequence are the infimum of positive number $\mathcal{K}$ for which :-

$$\sum_{n=1}^{\infty} |\zeta_n|^{-\mathcal{K}} < \infty. \tag{2.75}$$

Let $\mathcal{N}(\mathcal{R})$ be the number of terms of this sequence enclosed within the circle $|Z| = \mathcal{R}$. [19],[20] We described the density of the sequence by the two non-negative numbers:

$$\varrho = \lim_{\mathcal{R}\to\infty} \sup \frac{\ln \mathcal{N}(\mathcal{R})}{\ln \mathcal{R}};$$
$$\overline{\delta} = \lim_{\mathcal{R}\to\infty} \sup \frac{\mathcal{N}(\mathcal{R})}{\mathcal{R}^{\rho}}; \qquad \underline{\delta} = \lim_{\mathcal{R}\to\infty} \inf \frac{\mathcal{N}(\mathcal{R})}{\mathcal{R}^{\rho}}. \tag{2.76}$$

In this cases we say that the density (ϱ, δ) is smaller than (ϱ', δ') , if $\varrho < \varrho'$ or if $\varrho = \varrho'$ and also $\delta < \delta'$ (in which case we do not compare δ, δ'). In general the analytic function knows that ,[51] $\varrho < \rho$ or both $\varrho = \rho$ and $\delta \leq \tau\rho$. The Bargmann function has growth ($\rho = 2$, and $\tau = 1/2$) and this reveal that $\varrho < 2$ or both $\varrho = 0, \delta \leq 1$. Accordingly the density of zeros of the Bargmann function is smaller that $(\varrho = 2, \delta = 1)$. We now consider various examples :

• The coherent states $|A>$ is represented by the Bargmann function :

$$g(z) = \exp(Az - \frac{|A|^2}{2}). \tag{2.77}$$

Which is of order $\rho = 1$ and type $\tau = |A|$.

• Another example is the number eigenstate $|n>$ represented by the function:

$$g(z) = \frac{z^n}{\sqrt{n!}}. \tag{2.78}$$

which is of order $\rho = 0$. In fact any superposition of a finite number eigenstates is represented by a (finite) polynomial, which is of order $\rho = 0$.

• The Bargmann function of the squeezed state $|A; r, \theta, \lambda\rangle$ is :

$$g(z) = (1 - |\tau|^2)^{1/4} \exp[\frac{\tau}{2} z^2 + \beta z + \lambda] \tag{2.79}$$

$$\tau = -\tanh(\frac{1}{2} r) \exp(-i\theta) \ , \ \beta = A(1 - |\tau|^2)^{1/2} \ , \ \lambda = -\frac{1}{2} \tau^* A^2 - \frac{1}{2} |A|^2.$$

It has growth with order $\rho = 2$ and type $\tau = \frac{1}{2} \tanh(\frac{1}{2} r)$.

An example of a state whose Bargmann function has growth with a given order ρ and given type τ (and which may or may not be integers) is :

$$|\rho, \tau\rangle = \sum_{N=0}^{\infty} f_N |N\rangle \tag{2.80}$$

$$f_N = K \frac{\tau^{\frac{N}{\rho}} (N!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho} + 1)} \tag{2.81}$$

$$f(z) = \sum_{N=0}^{\infty} f_N z^N (N!)^{\frac{-1}{2}}. \tag{2.82}$$

Where K is a normalization constant given by :

$$K = \left[\sum_{N=0}^{\infty} \frac{\tau^{\frac{2N}{\rho}} (N!)}{[\Gamma(\frac{N}{\rho} + 1)]^2} \right]^{\frac{-1}{2}} \tag{2.83}$$

Then we can write :

$$|\rho, \tau\rangle = K * \sum_{N=0}^{\infty} \frac{\tau^{\frac{N}{\rho}} (N!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho} + 1)} |N\rangle \tag{2.84}$$

where $0<\rho<2$ If we Inserting Eq (2.81) in equation Eq(2.82) we get the following:

$$f(z) = \sum_{N=0}^{\infty} K \frac{(\tau^{\frac{1}{\rho}} z)^N}{\Gamma(\frac{N}{\rho} + 1)} = K E_{\frac{1}{\rho}}(\tau^{\frac{1}{\rho}} z), \tag{2.85}$$

where K is finite when $0 \leq \rho<2$; and also when $\rho = 2$ and $\tau < \frac{1}{2}$. The Bargmann function of this state $K E_{\frac{1}{\rho}}(\tau^{\frac{1}{\rho}} z)$ where $E_{\frac{1}{\rho}}(\tau^{\frac{1}{\rho}} z)$ is the Mittag-Leffler function, defined later.

2.5 The Mittag-Leffler function as Bargmann function with fractional order

Many quantum states used in practice (eg. coherent states, squeezed states) have a Bargmann functions with integer order of growth .

In this section we used Mittag-Leffler function as Bargmann function as example where the order of growth is fractional. The zeros of polynomial approximations of the Mittag-Leffler function are introduced.

2.5.1 The Mittag-Leffler function

The Mittag-Leffler function is named after the great Swedish mathematician Gosta Magnus Mittag-Leffler (1846-1927). He has worked on the general theory of functions, studying the relationship between independent and dependent variables [72]. In $1902 - 1905$ he introduced five subsequent notes, $[ML1, ML2, ML3, ML4, ML5]$ [71] with another exceptional function, which is now well-know and a useful favourite for many applications. The direct generalization of the Mittag-Lefer function was proposed by Wiman in his work [74] on zeros of function which is defined by the series :

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \alpha{>}0, z \in \mathbb{C} \tag{2.86}$$

More generally, the Mittag-Leffler function with two parameters has the form:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \qquad \alpha{>}0, \beta, z \in \mathbb{C} \tag{2.87}$$

Here z is a complex variable and α, β are arbitrary positive constants. When $\beta = 1$, $E_{\alpha,1}(z) = E_{\alpha}(z)$, also $\Gamma(.)$ is a gamma function, $\alpha{>}0$. For $0{<}\alpha{<}1$, it interpolates between the exponential and a hyper-geometric function . For the first parameter α with positive real part and any complex value of the second parameter, the function $E_{\alpha,\beta}(z)$ is an entire function of the complex variable z. Kilbas et al. studied the generalized Mittag-Leffler function with three parameters [65]. This function was also introduced by T.R Prabhakar in 1971[79]:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k (\gamma)_k}{k!\Gamma(\alpha k + \beta)}, \qquad (\gamma)_k = \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)}, \tag{2.88}$$

where α, β andγ are arbitrary positive constants, and $(\gamma)_k$ is the Pochhammer symbol[66].
When $\gamma = 1$, $E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$, and when $\gamma = \beta = 1$ then $E^{1}_{\alpha,1}(z) = E_{\alpha}(z)$.

In some other applications, a generalized Mittag-Leffler function has four parameters, the following function was introduced by Dzherbashian [80], and is defined as follows :

$$E^{\gamma,\eta}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k (\gamma)_{k\eta}}{\Gamma(\alpha k + \beta) k!}, \tag{2.89}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\eta \in \mathbb{N}$. When $(\gamma)_0 = 1$ and $(\gamma)_k = \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)}$.
It is easily seen, by comparing the definitions that :

$$\begin{aligned} E^{\gamma,1}_{\alpha,\beta}(z) &= E^{\gamma}_{\alpha,\beta}(z) \\ E^{1}_{\alpha,\beta}(z) &= E_{\alpha,\beta}(z). \end{aligned} \tag{2.90}$$

In the next section we provide details of some special properties of the Mittag-Leffler function.

2.5.2 The analytic properties of the Mittag-Leffler function

The Mittag-Lefer function has been introduced to give an answer to a classical question of complex analysis, namely to describe the procedure of analytic continuation of power series outside the disc of their convergence. A description of the most important properties of this function can be found in the third volum of the Bateman project [67],[71],[72],[76],[78],[80]. In this case when using their series representations for some parameters. It is easy to see that:

$$E_{0,1}(z) = \frac{1}{1-z}, |z| < 1 \tag{2.91}$$

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \tag{2.92}$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{e^z - 1}{z} \tag{2.93}$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \frac{1}{z^2}[e^z - z - 1] \tag{2.94}$$

$$E_{2,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{2k!} = \cosh(\sqrt{z}) \tag{2.95}$$

$$E_{2,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(2k+2)} = \frac{sinh(\sqrt{z})}{\sqrt{z}} \tag{2.96}$$

$$E^{1}_{1,1}(z) = E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \tag{2.97}$$

$$E^{1,1}_{\alpha,1}(z) = E_{\alpha,1}(z) = E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \tag{2.98}$$

$$E^{1,1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) = E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{e^z - 1}{z}. \tag{2.99}$$

And in general we can show some of this function as follows:-

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = (1 + z + \frac{z^2}{2!} + ... + \frac{z^k}{k!}) = e^z \tag{2.100}$$

$$E_{2,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{2k!} = \sum_{k=0}^{\infty} \frac{(z^{\frac{1}{2}})^2 k}{2k!} = \cosh(\sqrt{z}) \tag{2.101}$$

$$\begin{aligned} E_{1,3}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)!} = \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!} = \\ & \frac{1}{z^2}[\frac{z^2}{2!} + \frac{z^3}{3!} + ... + z - z + 1 - 1] = \frac{1}{z^2}[e^z - z - 1] \end{aligned} \tag{2.102}$$

$$\begin{aligned} E_{1,m}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+m)} &= \sum_{k=0}^{\infty} \frac{z^k}{(k+(m-1))!} = \frac{1}{z^{m-1}} \sum_{k=0}^{\infty} \frac{z^{k+(m-1)}}{(k+(m-1))!} \\ &= \sum_{k=0}^{\infty} \frac{1}{z^{m-1}}[e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!}], \quad m = 1, 2, ... \end{aligned} \tag{2.103}$$

If $\alpha, \beta > 0$ then they takes the formula [75],[76].

$$z^t E_{\alpha,\beta+t\alpha}(z) = E_{\alpha,\beta}(z) - \sum_{m=0}^{t-1} \frac{z^m}{\Gamma(m\alpha+\beta)}, \qquad t \in \mathbb{N}. \tag{2.104}$$

We can prove that:

$$E_{\alpha,\beta}(z) - \sum_{m=0}^{t-1} \frac{z^m}{\Gamma(m\alpha+\beta)} = \sum_{m=t}^{\infty} \frac{z^m}{\Gamma(m\alpha+\beta)} \tag{2.105}$$

If we put m-t=k or m=k+r we get :

$$\sum_{m=t}^{\infty} \frac{z^m}{\Gamma(m\alpha+\beta)} = \sum_{k=0}^{\infty} \frac{z^{k+t}}{\Gamma(k\alpha+t\alpha+\beta)} = z^t E_{\alpha,\beta+t\alpha}(z) \tag{2.106}$$

In a special case for $t = 1, 2, 3$ we obtain the following consequence:

$$z^1 E_{\alpha,\beta+1\alpha}(z) = E_{\alpha,\beta}(z) - \frac{z^0}{\Gamma(0\alpha+\beta)} = E_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)}. \tag{2.107}$$

$$z^2 E_{\alpha,\beta+2\alpha}(z) = E_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)} - \frac{z}{\Gamma(\alpha+\beta)}. \tag{2.108}$$

$$z^3 E_{\alpha,\beta+3\alpha}(z) = E_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)} - \frac{z}{\Gamma(\alpha+\beta)} - \frac{z^2}{\Gamma(2\alpha+\beta)}. \tag{2.109}$$

2.5.3 The Mittag-Leffler function of growth with fractional order

In section 2.4.1 we provided some basic definitions of the growth of the entire function [[19],[20],[71],[81],[82],[86],[77]]. The order and type of entire functions are important quantities for characterizing the functions. For an entire function the order and type can be found by the following formulae:

$$\rho = \overline{\lim_{k\mapsto\infty}} \frac{k\log k}{\log \frac{1}{|\mathcal{C}_k|}} \tag{2.110}$$

$$(\tau e\rho)^{1/\rho} = \overline{\lim_{k\mapsto\infty}} \left(k^{1/\rho}(\mid \mathcal{C}_k \mid)^{k/2}\right). \tag{2.111}$$

We see that the generalized Mittag-Leffler functions $E(\alpha,\beta;z)$ is entire functions of the order $\frac{1}{\alpha}$ and type 1 for all $\beta \in \mathbb{C}$.
Indeed, by using the Stirling formula (which gives an approximate value for the gamma function).

$$\begin{aligned}
\mathcal{C}_k &= \Gamma(\alpha k+1) \sim \sqrt{2\pi}(\alpha k)^{\alpha k+0.5}e^{-\alpha k} \qquad (k\mapsto\infty)\\
\mathcal{C}_k &= \Gamma(\alpha k+\beta) \sim \sqrt{2\pi}(\alpha k)^{\alpha k+\beta-0.5}e^{-\alpha k} \qquad (k\mapsto\infty).
\end{aligned} \tag{2.112}$$

We obtain.

$$\begin{aligned}
\rho_{E_{\alpha,1}} &= \limsup_{k\mapsto\infty} \frac{\log k}{-\log|1/\Gamma(\alpha k+1)|^{1/k}} = 1/\alpha,\\
\rho_{E_{\alpha,\beta}} &= \limsup_{k\mapsto\infty} \frac{\log k}{-\log|1/\Gamma(\alpha k+\beta)|^{1/k}} = 1/\alpha.
\end{aligned} \tag{2.113}$$

And

$$
\begin{aligned}
\tau_{E_{\alpha,1}} &= \limsup_{k\mapsto\infty} \frac{k\alpha}{e} |\frac{1}{\Gamma(\alpha k+1)}|^{1/(k\alpha)} = 1 \\
\tau_{E_{\alpha,\beta}} &= \limsup_{k\mapsto\infty} \frac{k\alpha}{e} |\frac{1}{\Gamma(\alpha k+\beta)}|^{1/(k\alpha)} = 1.
\end{aligned} \tag{2.114}
$$

We shall denote as $H(\rho,\tau)$ the space of functions of an order not exceeding ρ; and a type not exceeding τ if of order ρ . Clearly $H(\rho,\tau)$ is subset of $H(\rho',\tau')$ if $\rho < \acute{\rho}$ and also if $\rho = \rho'$ and $\tau < \tau'$. We conclude that the Mittag-Leffler function is subspace of $H(1/\alpha, 1)$.

Indeed, let us consider the slightly more general function for two-parametric :

$$
E_{\alpha,\beta}(\tau^{\alpha} z) = \sum_{k=0}^{\infty} \frac{(\tau^{\alpha} z)^k}{\Gamma(\alpha k+\beta)} \tag{2.115}
$$

The coefficients in this form are:

$$
\mathcal{C}_k = \frac{\tau^{\alpha k}}{\Gamma(\alpha k+\beta)} \qquad (k = 0, 1, 2, ...) \tag{2.116}
$$

Where $0 < \alpha < \infty$, and $0 < \tau < \infty$ are arbitrary real constant and β is a complex parameter . According to the theory of an entire function Eq(2.115) has order $\rho = 1/\alpha$ and type τ for any β. In this case, we can say that the two-parametric Mittag-Leffler function has an order $\rho = 1/\alpha$ and type $\tau = 1$, for any value of the parameter $\beta \in \mathbb{C}$.

2.5.4 The zeros of polynomial approximations of the Mittag-Leffler function

An entire function of fractional order can have infinitely many zeros. Also there are entire function which have few zero or no zeroes , (eg..the exponential function [81],[84],[85],[86], [87], [88],[89]). The Mittag-Leffler function which was given in Equations (2.86) and,(2.87) is an entire function of order $\frac{1}{\alpha}$ Consequently, Mittag-Leffler function $E_{\alpha,\beta}(z)$ might have an infinite number of zeros with the possible exception when $\frac{1}{\alpha}$ is an integer. In this case, there may be a finite number of zeros, or an infinite number of zeros. We can show that with the exception of $\alpha = \beta = 1$,the Mittag-Leffler function has an infinite number of zeros[88].

The Mittag-Leffler function $E_{(1,1)}(z)$ is equal to the exponential function e^z and is only the function which has no zeros. In this section we calculate the zeros of polynomial approximations to $E_{\alpha,\beta}(z)$ using Eq (2.86) when $\beta = 1$ can be $2; 4; 6; ....$ we demonstrate this procedure numerically for α increasing from $1.4 < \alpha < 1.99$ and where z is real. In Fig (2.4),(2.5),and (2.6) we plot some curves of $E_{\alpha,1}(z)$. For example when the $\alpha = 1.422$,$E_{\alpha,1}(z)$ curve crosses the x-axis two times yielding two zeros,the next larger value of α when $\alpha = 1.591$,$E_{\alpha,1}(z)$ has four zeros, and when $\alpha = 1.75$,$E_{\alpha,1}(z)$ has six zeros.

The Mittag-Leffler function $E_{\alpha,\beta}(z)$, which is a generalization of the exponential and trigonometric functions, arises frequently in problems of fractional calculus and hence, to understand the theory of fractional differential equations, one needs to understand properties of this function. One property which is of interest is the nature of its zeros. The main results regarding zeros of

$E_{\alpha,\beta}(z)$ when α is a real number lying between 1 and 3 may be summarized as follows: When $1 < \alpha < 2$, there is a finite number (possibly zero) of real zeros and an infinity of complex zeros. When $2 < \alpha \leq 3$, there are a finite number (possibly zero) of complex zeros and an infinite number of real zeros. The number of complex zeros goes as $\log \beta$ and the complex zeros are contained in a small region near the origin[85].

Remark : We note that the polynomial approximation to the function might introduce extra (fictitious) zeros. For example, for $\alpha, \beta = 1$, $E_{1,1}(z) = \exp(z)$ has no real or complex zeros, but its polynomial approximation :

$$e^z = 1 + z + \frac{z^2}{2!} + ... + \frac{z^N}{N!} = \sum_{N=0}^{\infty} \frac{z^N}{N!}, \tag{2.117}$$

has N zeros.

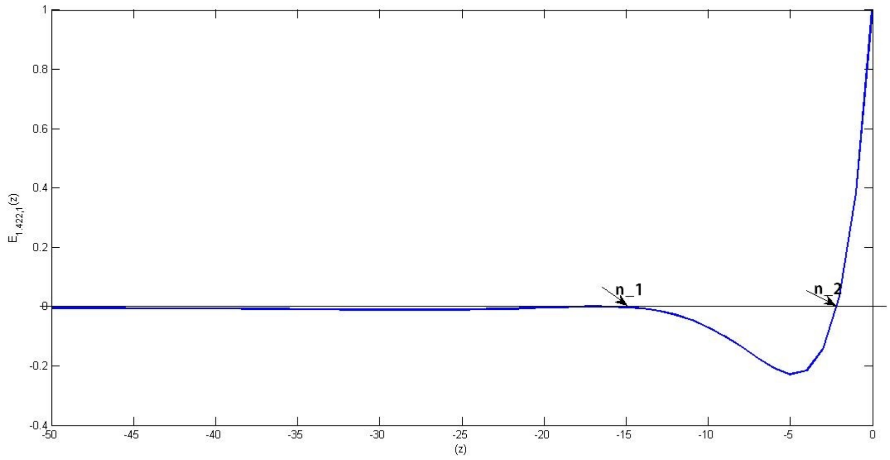

Figure 2.4: The zeros of the function $E_{\alpha,1}(z)$ for real z when $\alpha = 1.422$.

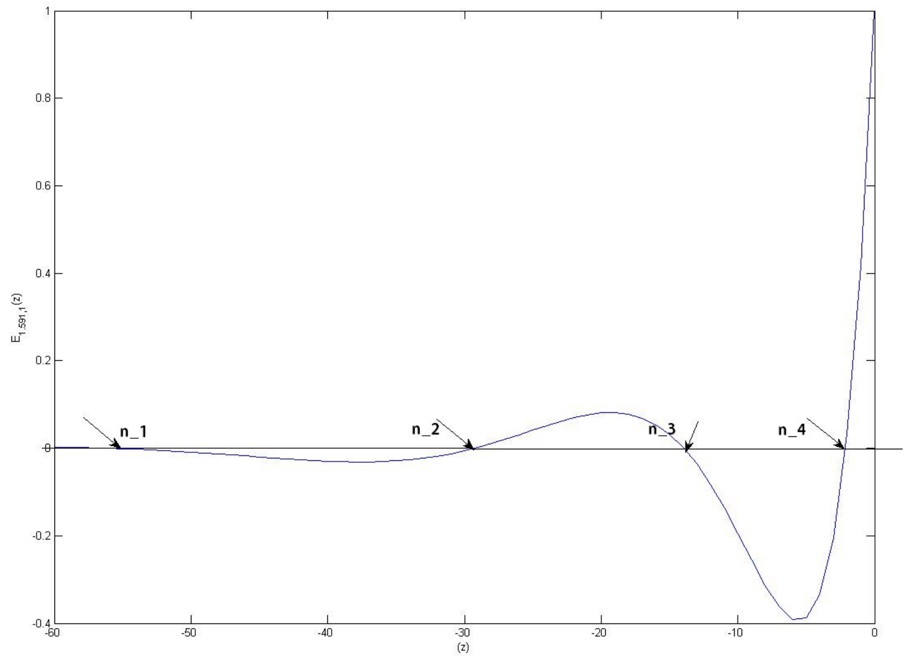

Figure 2.5: The zeros of the function $E_{\alpha,1}(z)$ for real z when $\alpha = 1.591$.

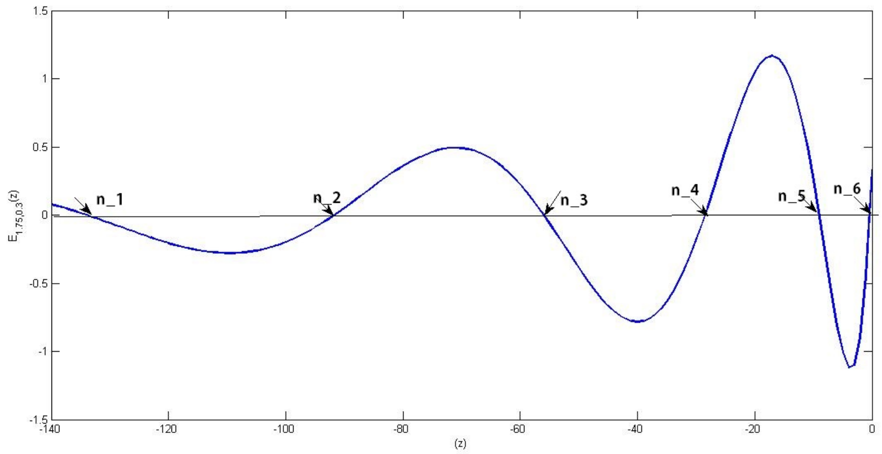

Figure 2.6: The zeros of the function $E_{\alpha,\beta}(z)$ for real z when $\alpha = \beta = 1.75$.

2.5.5 The Mittag-Leffler states: States with the Mittag-Leffler function as Bargmann function

In this section we extend the construction of the states which has the Bargmann function with a given order ρ (ρ can be any values between 0 and 2 , also it can be $\rho > 2$ but then the function is not normalizable) and given type σ by choosing the coefficients $\mathcal{C}_N$ in:

$$|\rho,\sigma\rangle = \sum_{N=0}^{\infty} \mathcal{G}_N|N\rangle; \qquad \mathcal{G}_N = K\mathcal{C}_N; \qquad \mathcal{C}_N = \frac{\sigma^{\frac{N}{\rho}}(N!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho}+\beta)}. \tag{2.118}$$

$$\mathcal{G}_N = K\frac{\sigma^{\frac{N}{\rho}}(N!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho}+1)}, \tag{2.119}$$

where K is a normalization constant given by the following:-

$$K = [\sum_{N=0}^{\infty} \frac{\sigma^{\frac{2N}{\rho}}(N!)}{[\Gamma(\frac{N}{\rho}+1)]^2}]^{\frac{-1}{2}}, \tag{2.120}$$

then we can write:

$$|\rho,\sigma\rangle = \sum_{N=0}^{\infty} \frac{\sigma^{\frac{N}{\rho}}(N!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho}+\beta)} * [\sum_{N=0}^{\infty} \frac{\sigma^{\frac{2N}{\rho}}(N!)}{\Gamma(\frac{N}{\rho}+\beta)}]^{\frac{-1}{2}}|N\rangle \tag{2.121}$$

K is finite when $0 \leq \rho<2$; and also when $\rho = 2$ and $\sigma < \frac{1}{2}$. The Bargmann function of this state $KE_{\frac{1}{\rho}}(\sigma^{\frac{1}{\rho}}z)$ where $E_{\frac{1}{\rho}}(\sigma^{\frac{1}{\rho}}z)$ is the Mittage-Leffer function. and we can writ it as follows:-

$$G(z) = \sum_{N=0}^{\infty} \mathcal{G}_N z^N (N!)^{\frac{-1}{2}}, \tag{2.122}$$

inserting $Eq(2.119)$ in equation $Eq(2.122)$ we get :

$$G(z) = \sum_{N=0}^{\infty} K \frac{\sigma^{\frac{N}{\rho}} (N!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho} + \beta)} z^N (N!)^{\frac{-1}{2}} = \sum_{N=0}^{\infty} K \frac{(\sigma^{\frac{1}{\rho}} z)^N}{\Gamma(\frac{N}{\rho} + \beta)} = K E_{\frac{1}{\rho}}(\sigma^{\frac{1}{\rho}} z). \quad (2.123)$$

We confine $z \in \mathbb{C}$ and consider the zeroes of the function $G(z)$ in Eq (2.123), where $E_{\frac{1}{\rho}}(\sigma^{\frac{1}{\rho}} z)$ is the Mittag-Leffler function when $\beta = 1$. As an example we present extensive numerical calculation of the function $G(z)$ in the complex plan. We show a three-dimensional plot of the real and imaginary parts. ρ can take all values between 0 and 2. We have considered numerical results are presented in figures $(2.7 - 2.10)$. The total numbers of the zeros in this case can be easily enumerated. In the special case where $\rho = 1$ the state (2.121) is reduced to usual coherent states and when $\rho = 2$ the state (2.121) is reduced to squeezed states .

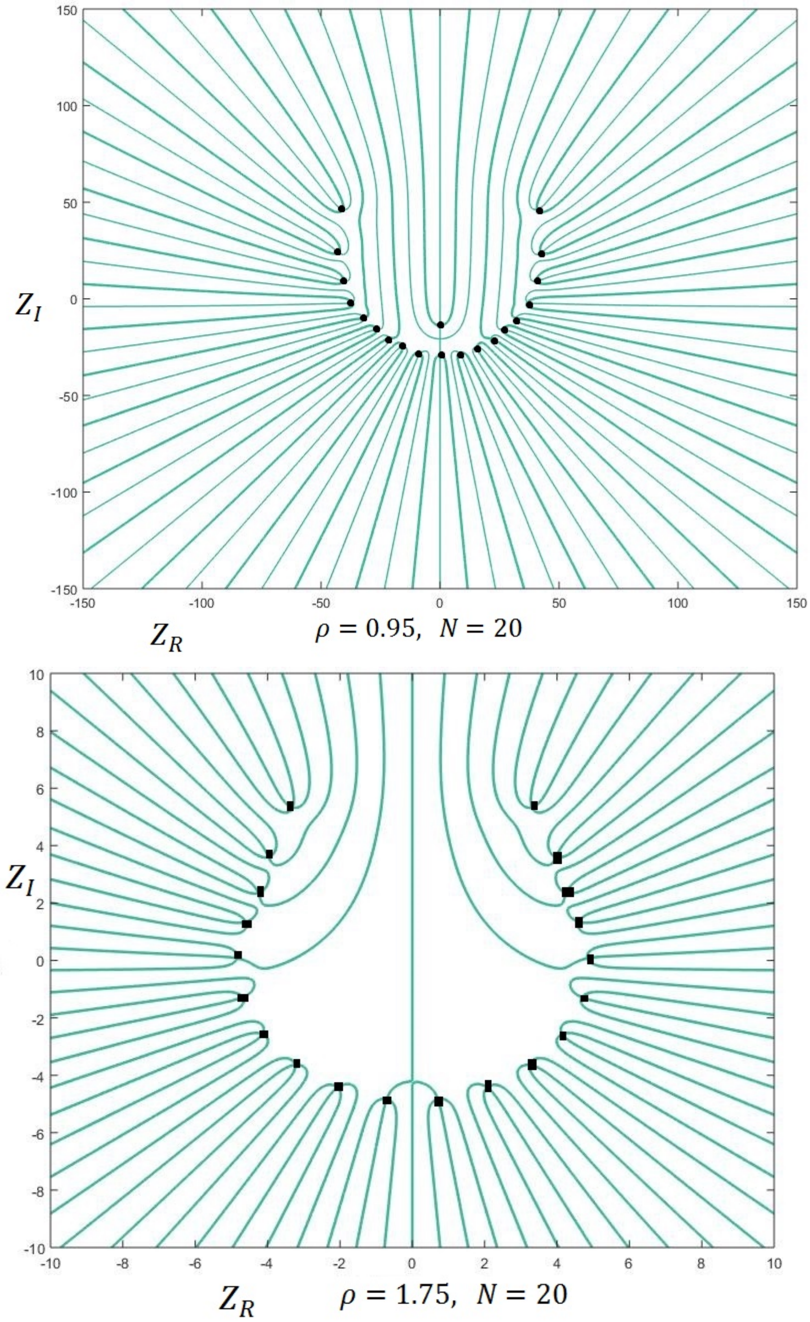

Figure 2.7: The zeros of the function $G_N(z)$ in Eq (2.123), with $N = 20$ for $E_{0.95}(z)$(top), and $E_{1.75}(z)$(bottom), Polynomial approximation have been made, and their zeros may be fictitious.

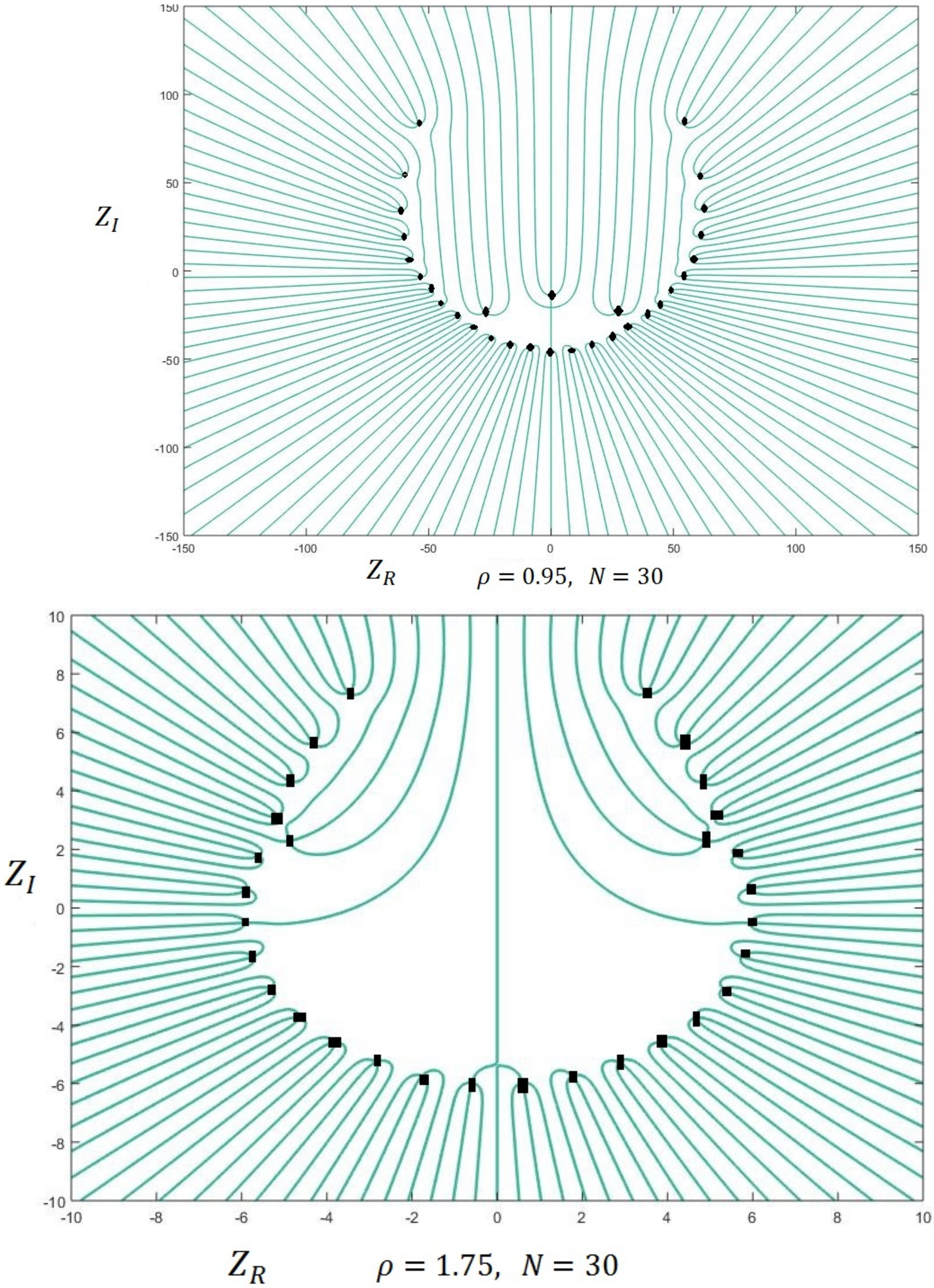

Figure 2.8: The zeros of the function $G_N(z)$in Eq (2.123), with $N = 30$ for $E_{0.95}(z)$(top), and $E_{1.75}(z)$(bottom), Polynomial approximation have been made, and their zeros may be fictitious.

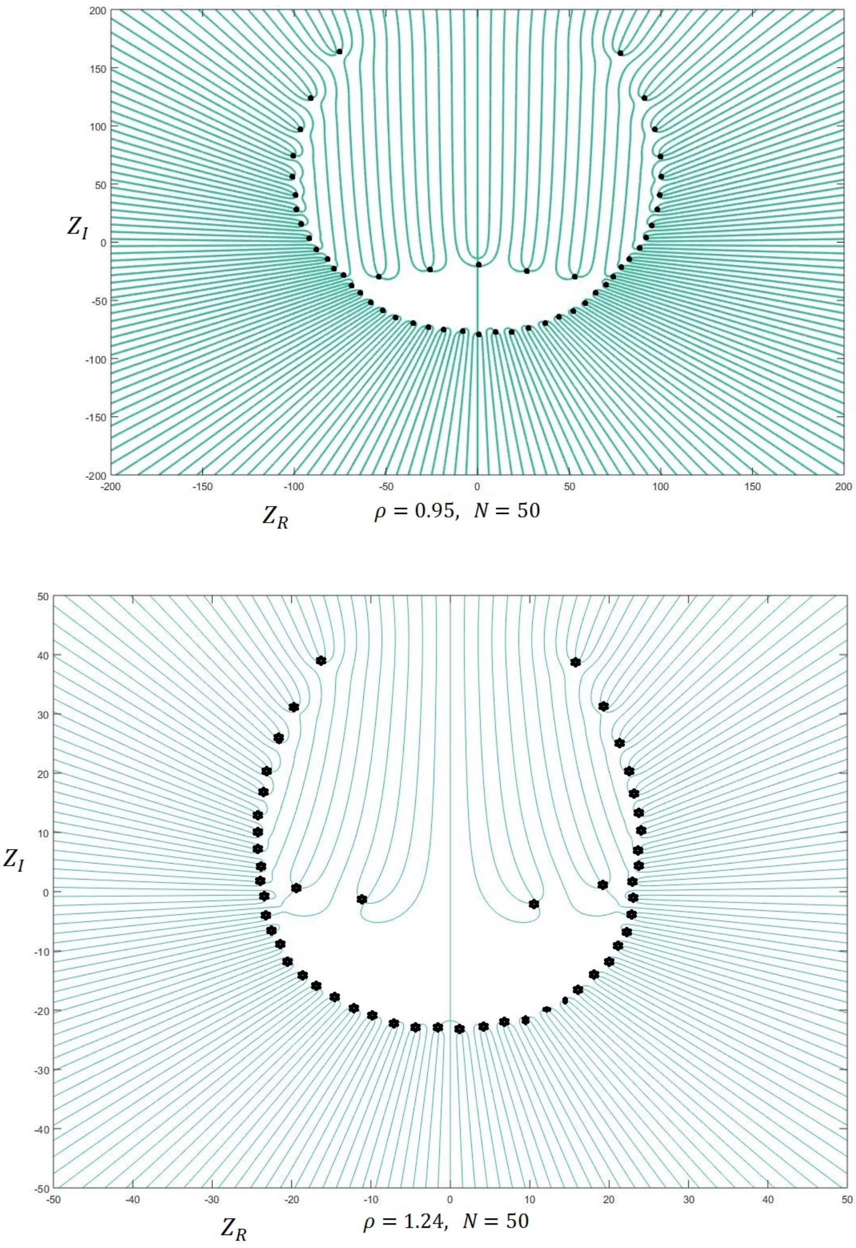

Figure 2.9: The zeros of the function $G_N(z)$ in Eq (2.123), with $N = 50$ for $E_{0.95}(z)$ (top),and $E_{1.24}(z)$ (bottom), Polynomial approximation have been made, and their zeros may be fictitious.

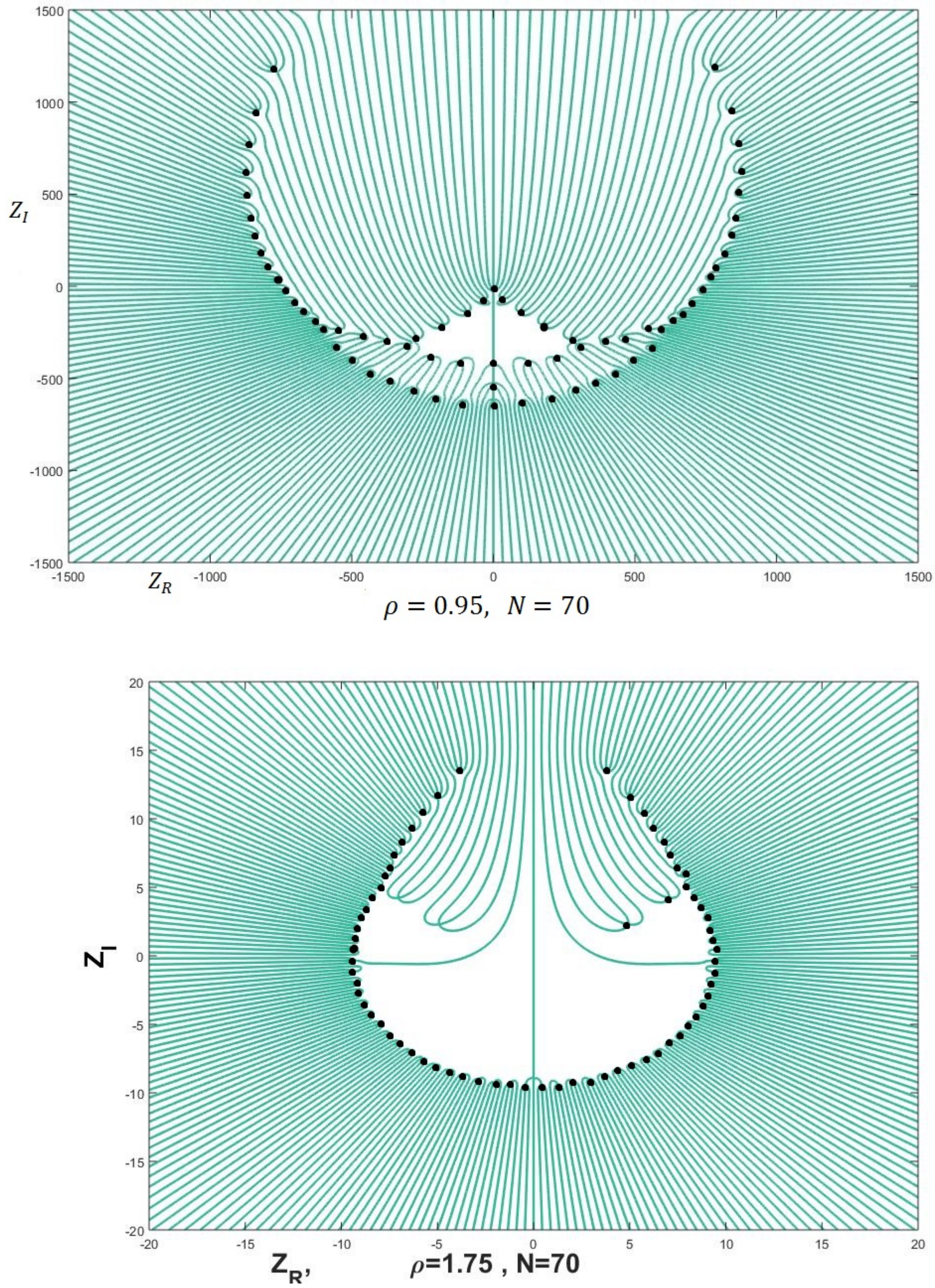

Figure 2.10: The zeros of the function $G_N(z)$ in Eq (2.123), with $N = 70$ for $E_{0.95}(z)$ (top), and $E_{1.75}(z)$ (bottom), Polynomial approximation have been made, and their zeros may be fictitious.

2.5.6 Quantum statistical properties of the Mittag-Leffler states

We studied the uncertainty properties of the state which has the Bargmann state, using the Mittag-Leffler state with a given order ρ and type σ. We start by calculating the position and the momentum uncertainty .The more accurately we know one of these values, the less accurately we know the other. Multiplying together the errors in the measurements of these values has to give a number greater than or equal to half of a constant, we can see that using the definition.

$$\Delta x = [< x^2 > - < x >^2]^{\frac{1}{2}}$$
$$< x^i >= \sum_{N,M} \mathcal{G}_N \mathcal{G}_M^* < N|(\frac{a + a^\dagger}{\sqrt{2}})^i|M > \tag{2.124}$$

Similarly, we have:

$$\Delta p = [< p^2 > - < p >^2]^{\frac{1}{2}}$$
$$< p^i >= \sum_{N,M} \mathcal{G}_N \mathcal{G}_M^* < N|(i\frac{a - a^\dagger}{\sqrt{2}})^i|M >, \tag{2.125}$$

and the uncertainty products are $\Delta x \Delta p$:

$$\Delta x \Delta p \geq \frac{1}{2} \tag{2.126}$$

We can note that:

$$a|n >= \sqrt{n}|n-1 > \qquad a^\dagger|n >= \sqrt{n+1}|n+1 > \tag{2.127}$$

Using Eq (2.119) and (2.120) we can find:

$$< x >= \sum_{N,M} \mathcal{G}_N \mathcal{G}_M^* < N|\frac{a + a^\dagger}{\sqrt{2}}|M > \tag{2.128}$$

$$< x^2 >= \sum_{N,M} \mathcal{G}_N \mathcal{G}_M^* < N|(\frac{a + a^\dagger}{\sqrt{2}})^2|M > .$$

For:

$$< N|\frac{a + a^\dagger}{\sqrt{2}}|M >= \frac{1}{\sqrt{2}}[< N|a|M > + < N|a^\dagger|M >] = \frac{1}{\sqrt{2}}[\sqrt{M}\delta_{N,M-1} + \sqrt{M+1}\delta_{N,M+1}]. \tag{2.129}$$

Then we have :

$$< x >= \sum_{N,M} \mathcal{G}_N \mathcal{G}_M^* \frac{1}{\sqrt{2}}[\sqrt{M}\delta_{N,M-1} + \sqrt{M+1}\delta_{N,M+1}] \tag{2.130}$$

$$< x >= \frac{1}{\sqrt{2}} \sum_{N,M} K^2 \frac{\sigma^{\frac{N}{\rho}}(N!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho}+1)} * \frac{\sigma^{\frac{M}{\rho}}(M!)^{\frac{1}{2}}}{\Gamma(\frac{M}{\rho}+1)}[\sqrt{M}\delta_{N,M-1} + \sqrt{M+1}\delta_{N,M+1}]. \tag{2.131}$$

$$< x >= \frac{1}{\sqrt{2}}K^2[\sum_{N=0} \frac{\sigma^{\frac{N}{\rho}}(N!)^{\frac{1}{2}}\sigma^{\frac{N+1}{\rho}}((N+1)!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho}+1)\Gamma(\frac{N+1}{\rho}+1)}\sqrt{N+1} + \sum_{M=0} \frac{\sigma^{\frac{M+1}{\rho}}(M!)^{\frac{1}{2}}\sigma^{\frac{M}{\rho}}((M+1)!)^{\frac{1}{2}}}{\Gamma(\frac{M}{\rho}+1)\Gamma(\frac{M+1}{\rho}+1)}\sqrt{M+1}]. \tag{2.132}$$

$$< x^2 >= \sum_{N,M} \mathcal{G}_N \mathcal{G}_M^* < N|(\frac{a+a^\dagger}{\sqrt{2}})^2|M > . \tag{2.133}$$

$$\begin{aligned} < x^2 >= \frac{1}{\sqrt{2}} K^2 [\sum_{N=0} \frac{\sigma^{\frac{N}{\rho}} (N!)^{\frac{1}{2}} \sigma^{\frac{N+2}{\rho}} ((N+2)!)^{\frac{1}{2}}}{\Gamma(\frac{N}{\rho}+1)\Gamma(\frac{N+2}{\rho}+1)} \sqrt{(N+1)(N+2)} \\ + \sum_{M=0} \frac{\sigma^{\frac{M+2}{\rho}} (M!)^{\frac{1}{2}} \sigma^{\frac{M}{\rho}} ((M+2)!)^{\frac{1}{2}}}{\Gamma(\frac{M}{\rho}+1)\Gamma(\frac{M+2}{\rho}+1)} \sqrt{(M+1)(M+2)} \\ + \sum_{N=0} \frac{\sigma^{\frac{2N}{\rho}} (N!)}{[\Gamma(\frac{N}{\rho}+1)]^2} (2N+1)]. \end{aligned} \tag{2.134}$$

$$\Delta x = [< x^2 > - < x >^2]^{\frac{1}{2}}. \tag{2.135}$$

Then we continue with the momentum uncertainty we have:

$$< p >= \sum_{N,M} \mathcal{G}_N \mathcal{G}_M^* < N|i\frac{a-a^\dagger}{\sqrt{2}}|M > \tag{2.136}$$

$$< p^2 >= \sum_{N,M} \mathcal{G}_N \mathcal{G}_M^* < N|(i\frac{a-a^\dagger}{\sqrt{2}})^2|M > .$$

And we have :

$$\Delta p = [< p^2 > - < p >^2]^{\frac{1}{2}} \tag{2.137}$$

Constructing the uncertainty from Δx and Δp we find that $\Delta x \Delta p$ is slightly higher then (0.5). Numerical results presented in Fig $(2.11 - 2.13)$. It is seen for ρ less than (1.6). The uncertainty products are $\Delta x \Delta p$, which takes values very close to (0.5). In this case the uncertainty Δx and Δp display some modest squeezing around $\rho = 0.7$ (i.e $\Delta x < 2^{-0.5}$).

Also we have calculated the average number of photons $< N >$ and the second-order correlation $g^(2)$.

$$g^(2) = \frac{< N^2 > - < N >}{< N >^2} \tag{2.138}$$

$$< N >= \sum_{N=0}^{\infty} N|\mathcal{G}_N|^2, \quad < N^2 >= \sum_{N=0}^{\infty} N^2|\mathcal{G}_N|^2 \tag{2.139}$$

$$\Delta N =< N^2 > - < N >^2 \tag{2.140}$$

The infinite sums have been truncated at $N = 15$, and the region of ρ has been limited to 0.1<ρ<2. We checked the average number of photons $< N >$ in all cases, We considered the numerical results that are presented in figures $[2.14 - 2.16]$, and the second-order correlation in figure (2.17).

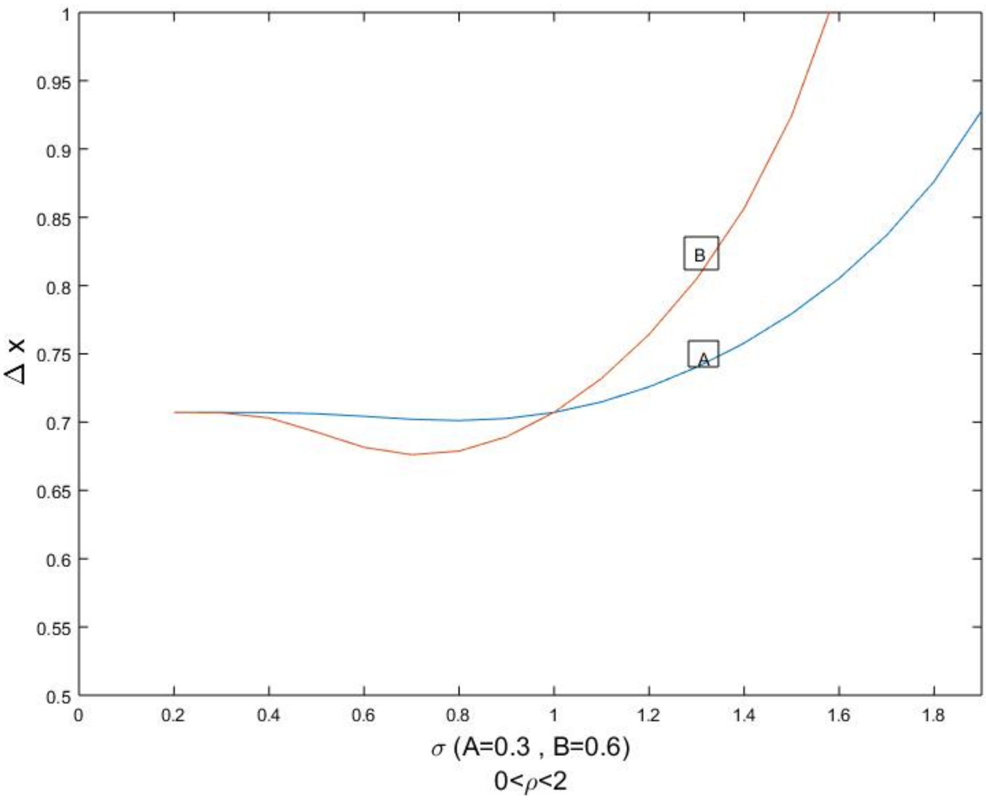

Figure 2.11: Δx for the state (2.121), as function of ρ for (A)$\sigma = 0.3$; and (B)$\sigma = 0.6$.

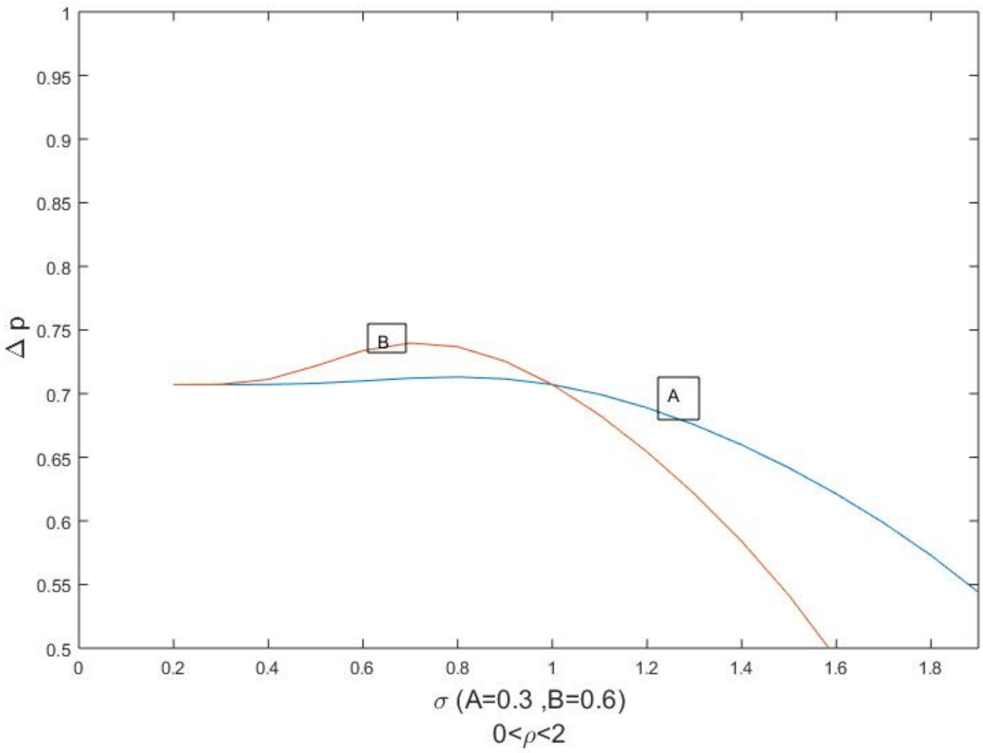

Figure 2.12: Δp for the state (2.121), as function of ρ for (A) $\sigma = 0.3$; and (B) $\sigma = 0.6$.

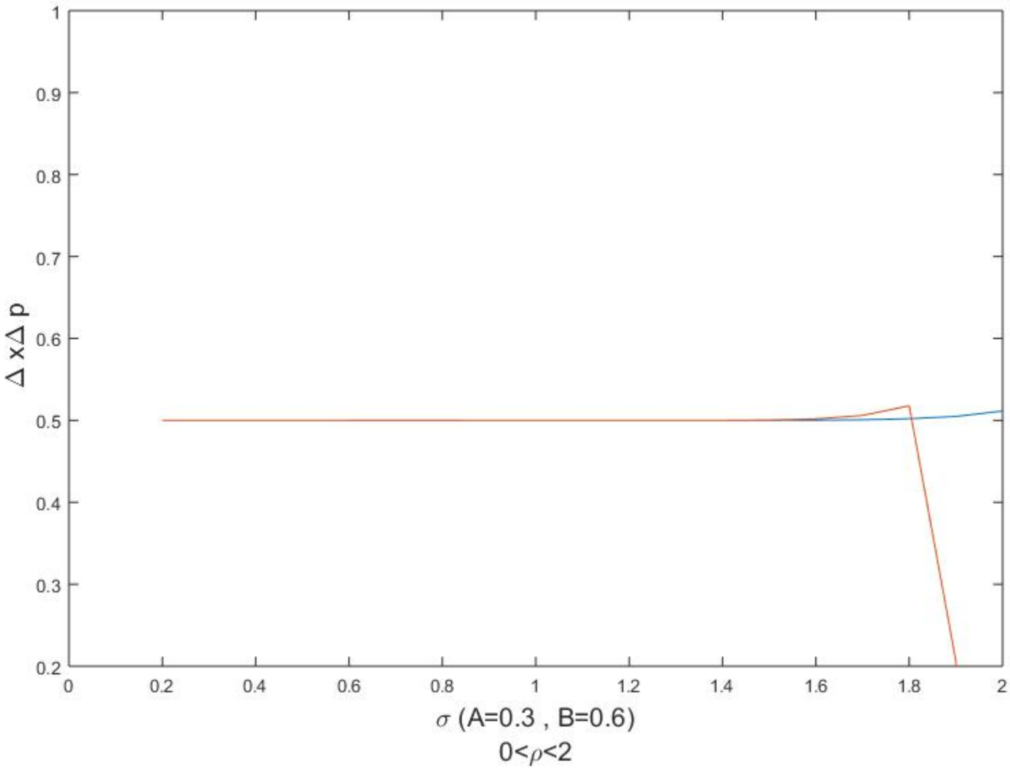

Figure 2.13: $\Delta x \Delta p$ for the state (2.121), as function of ρ for (A)$\sigma = 0.3$; and (B)$\sigma = 0.6$.

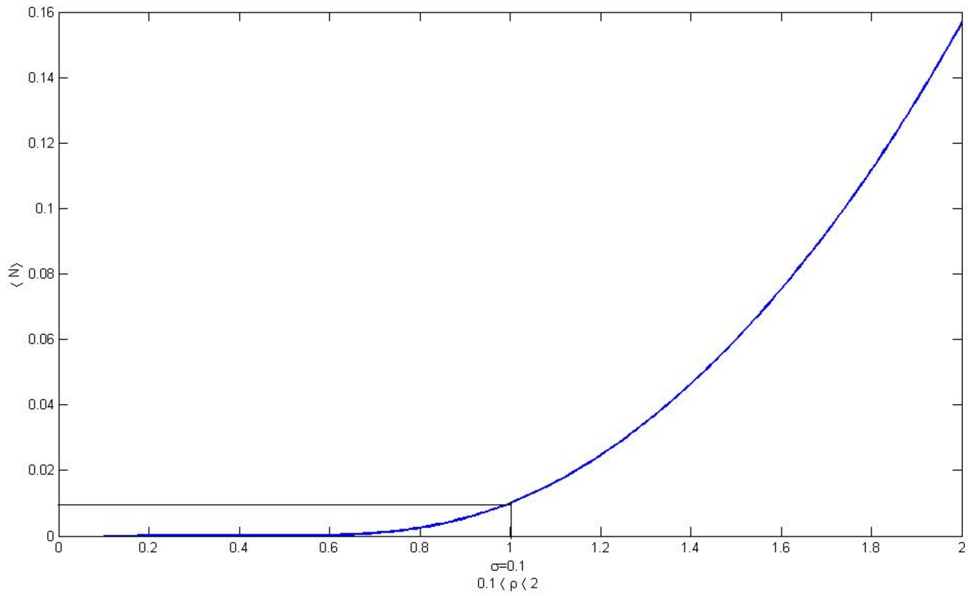

Figure 2.14: the average number of photon$< N >$ for the states Eq(2.121) (N=15), as a function of ρ and $\sigma = 0.1$.

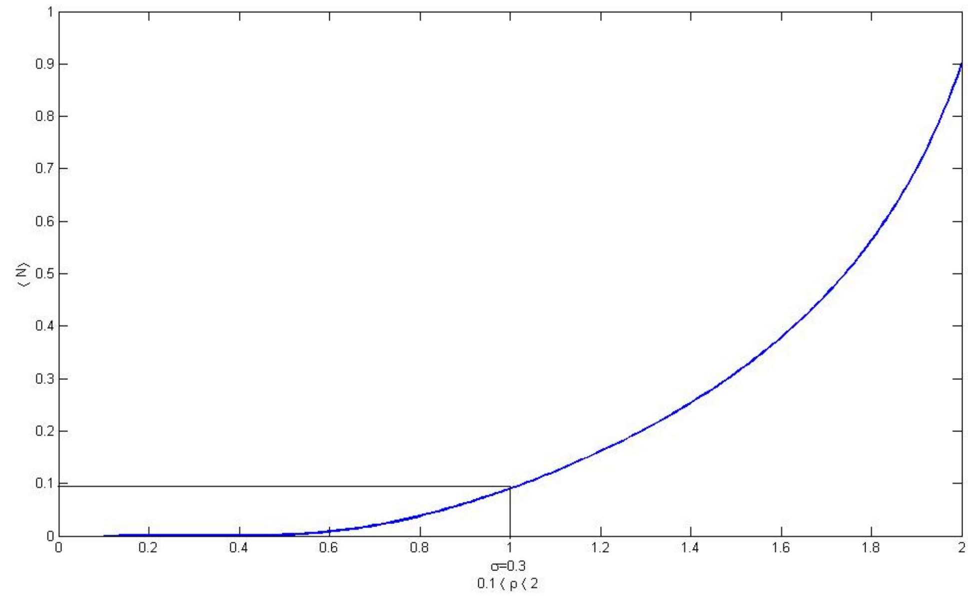

Figure 2.15: the average number of photon$< N >$ for the states Eq(2.121) (N=15), as a function of ρ and $\sigma = 0.3$.

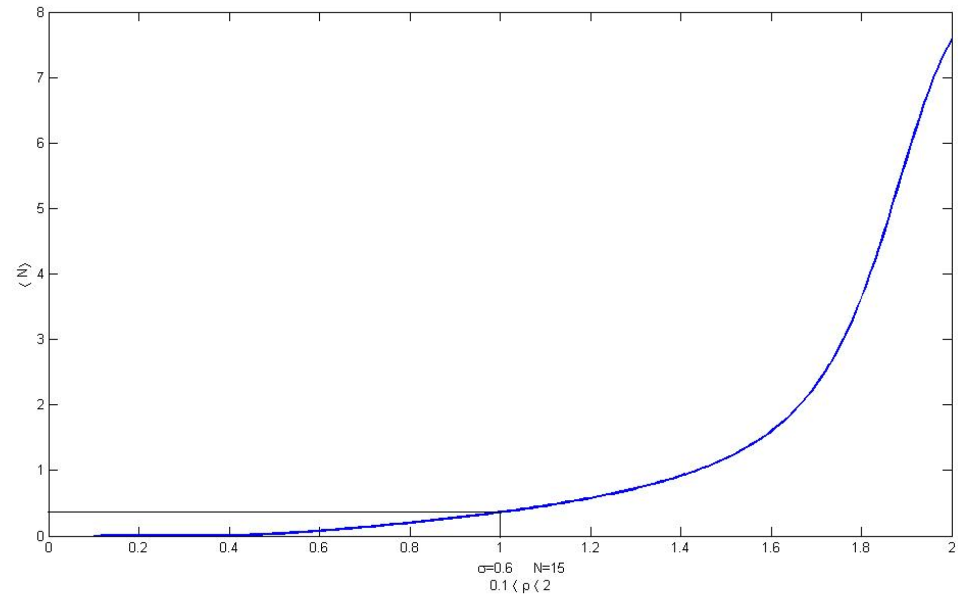

Figure 2.16: the average number of photon$< N >$ for the states Eq(2.121) (N=15), as a function of ρ and $\sigma = 0.6$.

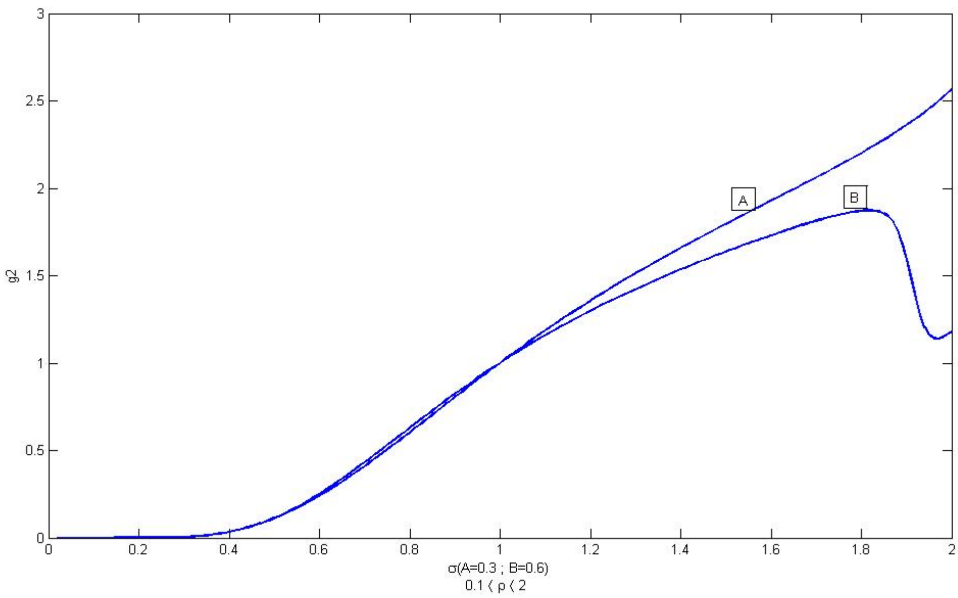

Figure 2.17: g^2 for the states Eq(2.121), as a function of ρ and (A)$\sigma = 0.3$, (B) $\sigma = 0.6$.

2.6 Conclusion

In this chapter, we stated the phase space methods for quantum particles on a real line $\mathcal{R}$. Therefore, the position and momenta belongs to $\mathcal{R}$, Some special states such as number states, displacement parity operators and coherent states are studied. The Wigner function and the Weyl function are of the most famous functions used in phase-space methods are defined. Also The Bargmann analytic representation in the complex plane, which was defined by the coherent states, and the growth of the Bargmann analytic function was considered. We also considered the growth of the Mittag-Leffler function, and the parametric and analytic properties of the function . We also considered the zeros of polynomial approximation to Mittag-Leffler function when $1 < \alpha < 2$. Furthermore, we considered the Bargmann function of the zeros of the function and we calculated the number of zeros for any value of α in the area of $0 < \alpha < 2$.

Chapter 3

Finite quantum systems

3.1 Introduction

Finite quantum systems were studied originally by Weyl and Schwinger, [27], [34]. Review with many references are given in the following [21],[23], [35],[37], [38], [39], [41],[43],[47]. In this chapter, finite systems in d-dimensional Hilbert space (both position and momentum states) are labelled by elements in $\mathbb{Z}_d$ where $\mathbb{Z}_d$ is the set of integers modulo d,(when d is an odd or even integer number). Some definitions are given in finite Hilbert space which will be used at a later point. In section 3.2, we study the position and momentum operators and Fourier transform in toroidal lattice $\mathbb{Z}_d \times \mathbb{Z}_d$. In section (3.3) we introduce the displacement operators in finite quantum systems. The Wigner and Weyl function in this phase space is defined in Section (3.4). Analytic representation is given in section (3.5). We conclude the chapter in Section (3.6)

3.2 Position and momentum states and Fourier transform

We introduced a quantum system with a d-dimensional Hilbert space $\mathcal{H}$, where position and momentum take values in $\mathbb{Z}_d$. We use the notation $|\mathcal{G}\rangle$ for the states in this particular Hilbert space $\mathbf{L}^2(\mathbb{Z}_d)$. We denote the position and momentum states by $|n\rangle_x$and $|n\rangle_p$ where the values of $n \in \mathbb{Z}_d$ form an orthonormal basis in $\mathbf{L}^2(\mathbb{Z}_d)$ They obey the following relations.

$$\begin{aligned} \langle X, m|X, n\rangle &= \delta(m, n), \\ \langle P, m|P, n\rangle &= \delta(m, n) \end{aligned} \tag{3.1}$$

where $m, n \in \mathbb{Z}_d$ and $\delta(m, n)$ is the Kronecker delta. Also they obey the completeness relation.

$$\sum_{n=0}^{d-1} |X, n\rangle\langle X, n| = \sum_{n=0}^{d-1} |P, n\rangle\langle P, n|) = 1 \tag{3.2}$$

The position and momentum states are related to each other through the finite Fourier transformation.

$$\mathcal{F} = d^{-1/2} \sum_{n=0}^{d-1} \omega(mn)|X, n\rangle \tag{3.3}$$

Where

$$\omega(\alpha) = exp(\frac{2\pi i\alpha}{d})$$

The Fourier transform is a unitary operator so

$$\mathcal{F}\mathcal{F}^{\dagger} = \mathcal{F}^{\dagger}\mathcal{F} = 1,$$
$$\mathcal{F}^{4} = 1 \tag{3.4}$$

Equation 3.4 implies that the Fourier operator has only four distinct eigenvalues namely $1, -1, i, -i$ with certain multiplicity [49]. Using the Fourier transform we define another orthonormal basis, the momentum states as:

$$|P, n\rangle = \mathcal{F}|X, n\rangle = d^{-1/2}\sum_{n=0}^{d-1}\omega(mn)|X, n\rangle \tag{3.5}$$

The position operator $\hat{x}$ and the momentum operator $\hat{p}$ are defined as:

$$\hat{x} = \sum_{n=0}^{d-1} n|X, n\rangle\langle X, n|;$$
$$\hat{p} = \sum_{n=0}^{d-1} n|P, n\rangle\langle P, n| \tag{3.6}$$

It is easily seen that:

$$\mathcal{F}\hat{x}\mathcal{F}^{\dagger} = \hat{p};$$
$$\mathcal{F}\hat{p}\mathcal{F}^{\dagger} = -\hat{x} \tag{3.7}$$

3.3 Displacement operators

The position states and momentum states in finite quantum systems are labelled by elements in $\mathbb{Z}_d$ where d is the dimension system. The phase space is the toroidal lattice $\mathbb{Z}_d \times \mathbb{Z}_d$. In this case we defined the displacement operators as:

$$\begin{aligned}\mathfrak{Z} = \exp[i\frac{2\pi}{d}\hat{x}] = \sum_{n=0}^{d-1} \exp\left[\frac{2\pi}{d}n\right] |X, n\rangle\langle n, X| = \omega(n);\\ \mathfrak{X} = \exp[-i\frac{2\pi}{d}\hat{p}] = \sum_{n=0}^{d-1} \exp\left[\frac{2\pi}{d}n\right] |P, n\rangle\langle n, P| = \omega(-n)\end{aligned} \tag{3.8}$$

$$\begin{aligned}\mathfrak{Z}^d = \mathfrak{X}^d = 1\\ \mathfrak{X}^\beta\mathfrak{Z}^\alpha = \mathfrak{Z}^\alpha\mathfrak{X}^\beta\omega(-\alpha\beta)\end{aligned} \tag{3.9}$$

Where α, β are integers in $\mathbb{Z}_d$. These operators perform displacements along the X and P axes in the $\mathbb{Z}_d \times \mathbb{Z}_d$ phase-space obeying the following relations.

$$\begin{aligned}\mathfrak{Z}^\alpha|P; n\rangle = |P; n+\alpha\rangle; \qquad \mathfrak{Z}^\alpha|X; n\rangle = \omega(\alpha n)|X, n\rangle;\\ \mathfrak{X}^\beta|P; n\rangle = \omega(-\beta n)|P, n\rangle; \qquad \mathfrak{X}^\beta|X; n\rangle = |X; n+\beta\rangle\end{aligned} \tag{3.10}$$

We define the general displacement operators as:

$$\begin{aligned}\mathcal{D}(\alpha, \beta) = \mathfrak{Z}^\alpha\mathfrak{X}^\beta\omega(-2^{-1}\alpha\beta) = \mathfrak{X}^\beta\mathfrak{Z}^\alpha\omega(2^{-1}\alpha\beta);\\ (\mathcal{D}(\alpha, \beta))^\dagger = \mathcal{D}(-\alpha, -\beta).\end{aligned} \tag{3.11}$$

Which have the following properties:

$$[\mathcal{D}(\alpha,\beta)]^{\dagger} = \mathcal{D}(-\alpha,-\beta)$$
$$[\mathcal{D}(\alpha,\beta)]^{\dagger}\,\mathcal{D}(\alpha,\beta) = I. \tag{3.12}$$

Using Eq (3.4) we can prove the multiplication rule [47],[54].

$$\mathcal{D}(\alpha,\beta)\mathcal{D}(\alpha_1,\beta_1) = \mathcal{D}(\alpha+\alpha_1,\beta+\beta_1)\ \omega[2^{-1}(\alpha\beta_1-\beta\alpha_1)] \tag{3.13}$$

It is easy to see that.

$$\mathcal{D}(\alpha,\beta)|X;n\rangle = \omega(2^{-1}\alpha\beta+\alpha n)|X;n+\beta\rangle;$$
$$\mathcal{D}(\alpha,\beta)|P;n\rangle = \omega(-2^{-1}\alpha\beta-\beta n)|P;n+\alpha\rangle; \tag{3.14}$$

We can prove that using Eq (3.10) so if :

$$\mathcal{D}(\alpha,\beta) = \mathfrak{Z}^{\alpha}\mathfrak{X}^{\beta}\omega(-2^{-1}\alpha\beta)$$
$$\mathfrak{Z}^{\alpha}\mathfrak{X}^{\beta}|X;n\rangle = \omega(\alpha(n+\beta))|X;n+\beta\rangle \tag{3.15}$$

Then :

$$\mathcal{D}(\alpha,\beta)|X;n\rangle = \mathfrak{Z}^{\alpha}\mathfrak{X}^{\beta}\omega(-2^{-1}\alpha\beta)|X;n\rangle$$
$$\omega(\alpha(n+\beta))\omega(-2^{-1}\alpha\beta)|X;n+\beta\rangle = \omega(\alpha n+\alpha\beta-2^{-1}\alpha\beta)|X;n\rangle \tag{3.16}$$

Therefore,

$$\mathcal{D}(\alpha,\beta)|X;n\rangle = \omega(2^{-1}\alpha\beta + \alpha n)|X;n+\beta\rangle. \tag{3.17}$$

Also we can use the definitions of the operators $\mathfrak{Z}$ and $\mathfrak{X}$ to prove that:-

$$\begin{aligned} \mathcal{D}(\alpha,\beta)x[\mathcal{D}(\alpha,\beta)]^{\dagger} &= x - \beta I, \\ \mathcal{D}(\alpha,\beta)p[\mathcal{D}(\alpha,\beta)]^{\dagger} &= p - \alpha I. \end{aligned} \tag{3.18}$$

Acting with the Fourier operator on the displacement operators we get the following:

$$\begin{aligned} \mathcal{F}\mathfrak{X}\mathcal{F}^{\dagger} &= \mathfrak{Z}\ ;\ \mathcal{F}\mathfrak{Z}\mathcal{F}^{\dagger} = \mathfrak{X}^{-1} \\ \mathcal{F}\mathcal{D}(\alpha,\beta)\mathcal{F}^{\dagger} &= \mathcal{D}(\beta,-\alpha). \end{aligned} \tag{3.19}$$

The marginal properties of the displacement operator in finite systems give the following:-

$$\begin{aligned} \frac{1}{d}\sum_{\alpha=0}^{d-1}\mathcal{D}(\alpha,\beta) &= |X;2^{-1}\beta\rangle\langle X;-2^{-1}\beta| \\ \frac{1}{d}\sum_{\beta=0}^{d-1}\mathcal{D}(\alpha,\beta) &= |P;2^{-1}\alpha\rangle\langle P;-2^{-1}\alpha| \\ \frac{1}{d}\sum_{\alpha\beta=0}^{d-1}\mathcal{D}(\alpha,\beta) &= P(0,0). \end{aligned} \tag{3.20}$$

The first two properties are proved by multiplying both sides by the momentum and position states. The third one is proved from the first one with the extra summation $\sum_{\beta=0}^{d-1}$.

3.4 Wigner and Weyl functions

In finite dimensional systems we define the Wigner function as follows:-

$$\mathcal{W}_{\Theta}(\alpha, \beta) = \mathcal{T}r[\Theta P(\alpha, \beta)], \tag{3.21}$$

where Θ is an arbitrary operator. The Wigner function can only take real values when the operator Θ is Hermitian,but for non-Hermitian operators it is complex.

It can also be defined as:-

$$\begin{aligned}
\mathcal{W}_{\Theta}(\alpha, \beta) &= \sum_{\mathcal{L}}^{d-1} \omega(2\alpha\beta - 2\alpha\mathcal{L})\Theta_x(\mathcal{L}, 2\beta - \mathcal{L}) \\
\mathcal{W}_{\Theta}(\alpha, \beta) &= \sum_{\mathcal{L}}^{d-1} \omega(2\beta\mathcal{L} - 2\alpha\beta)\Theta_p(\mathcal{L}, 2\alpha - \mathcal{L}).
\end{aligned} \tag{3.22}$$

The Weyl function is defined as:-

$$\begin{aligned}
\tilde{\mathcal{W}}_{\Theta}(\mu, \nu) &= \mathcal{T}r[\Theta \mathcal{D}(\mu, \nu)] \\
\tilde{\mathcal{W}}_{\Theta}(\mu, \nu) &= \sum_{\mathcal{L}}^{d-1} \omega(\mu\mathcal{L} + 2^{-1}\mu\nu)\Theta_x(\mathcal{L}, \nu + \mathcal{L}).
\end{aligned} \tag{3.23}$$

The marginal properties of the Wigner function for odd d -dimensional,[47] has shown:

$$\frac{1}{d}\sum_{\alpha=0}\mathcal{W}_{\Theta}(\alpha,\beta)=\Theta_x(\beta,\beta),$$
$$\frac{1}{d}\sum_{\beta=0}\mathcal{W}_{\Theta}(\alpha,\beta)=\Theta_p(\alpha,\alpha),$$
$$\frac{1}{d}\sum_{\alpha,\beta=0}\mathcal{W}_{\Theta}(\alpha,\beta)=\mathcal{T}r\Theta. \tag{3.24}$$

We can prove this property using Eq.(3.22) as:

$$\frac{1}{d}\sum_{\alpha=0}\mathcal{W}_{\Theta}(\alpha,\beta)=\Theta_x(\beta,\beta),$$
$$\frac{1}{d}\sum_{\alpha}\sum_{\mathcal{L}}^{d-1}\omega(2\alpha\beta-2\alpha\mathcal{L})\Theta_x(\mathcal{L},2\beta-\mathcal{L})=$$
$$\frac{1}{d}\sum_{\mathcal{L},\alpha}^{d-1}\omega(\alpha(2\beta-2\mathcal{L}))\Theta_x(\mathcal{L},2\beta-\mathcal{L})=\Theta_x(\beta,\beta). \tag{3.25}$$

In this case the Wigner function in Eq (3.24) can be read as the probability distribution of the particle in the position and the momentum phase space. The marginal properties of the wyel function are given as:

$$\frac{1}{d}\sum_{\mu=0}\tilde{\mathcal{W}}_{\Theta}(\mu,\nu)=\Theta_x(-2^{-1}\nu,2^{-1}\nu),$$
$$\frac{1}{d}\sum_{\nu=0}\tilde{\mathcal{W}}_{\Theta}(\mu,\nu)=\Theta_p(-2^{-1}\mu,2^{-1}\mu),$$
$$\frac{1}{d}\sum_{\mu,\nu=0}\tilde{\mathcal{W}}_{\Theta}(\mu,\nu)=\mathcal{W}_{\Theta}(0,0) \tag{3.26}$$

The Wigner function and Wyel function are related to each other through the Fourier transform.

$$\tilde{\mathcal{W}}_{\Theta}(\mu,\nu) = \frac{1}{d}\sum_{\alpha,\beta=0}\mathcal{W}_{\Theta}(\alpha,\beta)\omega(\mu\beta-\nu\alpha) \tag{3.27}$$

3.5 Analytic representation of finite quantum systems

Let $|X;m\rangle$,and $|P;m\rangle$ (where $m \in \mathbb{Z}(d)$) be the position and momentum bases which are related through a Fourier transform, as follows:

$$\begin{aligned} &|P;n\rangle = \mathcal{F}|X;n\rangle; \qquad \mathcal{F} = d^{-1/2}\sum_{m,n}\omega(mn)|X;m\rangle\langle X;n|; \\ &\omega(m) = \exp\left[i\frac{2\pi m}{d}\right] \end{aligned} \tag{3.28}$$

Let $|\mathcal{G}\rangle$ be an arbitrary pure normalised state :

$$|\mathcal{G}\rangle = \sum_{m}\mathcal{G}_m|X;m\rangle; \qquad \sum_{m}|\mathcal{G}_m|^2 = 1 \tag{3.29}$$

We will use the notation :

$$|\mathcal{G}^*\rangle = \sum_{m}\mathcal{G}_m^*|X;m\rangle; \;\; \langle\mathcal{G}| = \sum_{m}\mathcal{G}_m^*\langle X;m|; \;\; \langle\mathcal{G}^*| = \sum_{m}\mathcal{G}_m\langle X;m| \tag{3.30}$$

We define the analytic representation of state $|\mathcal{G}\rangle$,[51],[52],[57], [60],[61], [62], as follows:

$$
\begin{aligned}
G(z) &= [\mathcal{N}(z)]^{1/2} d^{1/2} \exp(\frac{-i}{2} z_I z) \langle z^* | \mathcal{G} \rangle \\
&= \pi^{-1/4} \sum_{m=0}^{d-1} \mathcal{G}_m \, \Theta_3 \left[\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}}; \frac{i}{d} \right]
\end{aligned} \tag{3.31}
$$

where Θ_3 , [63],[64], [60],[57] ,is the Theta function:

$$
\Theta_3(u, \tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi\tau n^2 + i2nu); \tag{3.32}
$$

and it has the following property

$$
\Theta_3(u, \tau) = (-i\tau)^{-\frac{1}{2}} \exp(\frac{u^2}{\pi i \tau}) \Theta_3[\frac{u}{\tau}, -\frac{1}{\tau}]. \tag{3.33}
$$

To prove this property we use the series

$$
\Theta_3(u, \tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi\tau n^2 + i2nu); \tag{3.34}
$$

And we use the relation :

$$
\sum_{n=-\infty}^{\infty} \exp(-\tau(u+n)^2) = \sqrt{\frac{\pi}{\tau}} \sum_{k=-\infty}^{\infty} \exp(2\pi iku - (\frac{\pi^2 k^2}{\tau})) \tag{3.35}
$$

(we assume $\tau = \frac{-\pi}{i\tau}$ and $u = \frac{u}{\pi}$)

Left part:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \exp(-\tau(u+n)^2) &= \sum_{n=-\infty}^{\infty} \exp(-\tau u^2 - \tau n^2 - 2un\tau) \\
&= \sum_{n=-\infty}^{\infty} \exp(\frac{\pi u^2}{\pi^2 i\tau} + \frac{\pi n^2}{i\tau} + \frac{2\pi un}{\pi i\tau}) \\
&= \sum_{n=-\infty}^{\infty} \exp(\frac{u^2}{\pi i\tau}) \exp(-\frac{i\pi n^2}{\tau} + \frac{2i(-u)n}{\tau}) \\
&= \exp(\frac{u^2}{i\pi\tau}) \sum_{n=-\infty}^{\infty} \exp(-\frac{i\pi n^2}{\tau} + \frac{2inu}{\tau}) \\
&= \exp(\frac{u^2}{i\pi\tau})\Theta_3(\frac{u}{\tau}; \frac{-1}{\tau}) \qquad (3.36)
\end{aligned}$$

Right part

$$\begin{aligned}
\sqrt{\frac{\pi}{\tau}} \sum_{k=-\infty}^{\infty} \exp(2\pi iku - (\frac{\pi^2 k^2}{\tau}) &= \sqrt{-\frac{i\pi\tau}{\pi}} \sum_{k=-\infty}^{\infty} \exp(\frac{2i\pi ku}{\pi} + i\pi\tau k^2) \\
&= \sqrt{-i\tau} \sum_{k=-\infty}^{\infty} \exp(2iku + i\pi\tau k^2) \\
&= \sqrt{-i\tau}\Theta_3(u;\tau) \qquad (3.37)
\end{aligned}$$

Left part = right part

$$\Theta_3(u,\tau) = (-i\tau)^{-\frac{1}{2}} \exp(\frac{u^2}{\pi i\tau})\Theta_3[\frac{u}{\tau}, -\frac{1}{\tau}]. \qquad (3.38)$$

The coefficients $\mathcal{G}_m$ in Eq.(3.29) are given by

$$\mathcal{G}_m = 2^{-1/2}\pi^{-3/4}d^{-3/2} \int_S d\mu(z)\Theta_3\left[\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}}; \frac{i}{d}\right] G(z^*). \qquad (3.39)$$

It is easy to see that :-

$$
\begin{aligned}
G(z+\sqrt{2\pi d}) &= G(z) \\
G(z+i\sqrt{2\pi d}) &= G(z)\exp\left(\pi d - iz\sqrt{2\pi d}\right).
\end{aligned} \tag{3.40}
$$

To prove the first one

$$
G(z+\sqrt{2\pi d}) = \pi^{-1/4}\sum_{m=0}^{d-1}\mathcal{G}_m\,\Theta_3\left[\frac{\pi m}{d}-(z+\sqrt{2\pi d})\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right]
$$

$$
\begin{aligned}
\Theta_3\left[\frac{\pi m}{d}-(z+\sqrt{2\pi d})\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right] &= \sum_{n=-\infty}^{\infty}\exp(-\frac{m^2\pi}{d}+2im(\frac{\pi m}{d}-z\sqrt{\frac{\pi}{2d}}-\sqrt{2\pi d}\sqrt{\frac{\pi}{2d}} \\
&= \sum_{n=-\infty}^{\infty}\exp(\frac{i^2m^2\pi}{d}+2im(\frac{\pi m}{d}-z\sqrt{\frac{\pi}{2d}}\exp(2im\pi) \\
&= \Theta_3(\frac{\pi m}{d}-z\sqrt{\frac{\pi}{2d}};\frac{i}{d})
\end{aligned} \tag{3.41}
$$

Then we can prove that:

$$
G(z+\sqrt{2\pi d}) = \pi^{-1/4}\sum_{m=0}^{d-1}\mathcal{G}_m\,\Theta_3\left[\frac{\pi m}{d}-z\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right] = G(z) \tag{3.42}
$$

in the same way we can prove the second one.

- Therefore it is sufficient to have this function $G(z)$ in a cell :

$$
S = [M\sqrt{2\pi d},(M+1)\sqrt{2\pi d})\times[N\sqrt{2\pi d},(N+1)\sqrt{2\pi d}) \tag{3.43}
$$

where (M, N) are integers labelling the cell.

• The scalar product is given by the following:

$$\langle f^*|\mathcal{G}\rangle = \frac{1}{d^{3/2}\sqrt{2\pi}} \int_S d\mu(z) F(z^*) G(z) = \sum f_m \mathcal{G}_m;$$
$$d\mu(z) = d^2 z \exp\left(-z_I^2\right). \tag{3.44}$$

These relations are proved using the orthogonality relation [51]

$$2^{-1/2}\pi^{-1}d^{-3/2} \int_S d\mu(z)\Theta_3\left[\frac{\pi n}{d} - z\sqrt{\frac{\pi}{2d}}; \frac{i}{d}\right] \times \Theta_3\left[\frac{\pi m}{d} - z^*\sqrt{\frac{\pi}{2d}}; \frac{i}{d}\right] = \delta(m, n) \tag{3.45}$$

where $m, n \in \mathbb{Z}_d$.

As special cases, we derive the analytic representation of the position states $|X; m\rangle$ and the momentum states $|P; m\rangle$ as follows :

$$|X; m\rangle \longrightarrow \pi^{-1/4}\ \Theta[\frac{\pi m}{d} - z(\frac{\pi}{2d})^{1/2}\ ; \frac{i}{d}] \tag{3.46}$$

And :

$$|P; m\rangle \longrightarrow \pi^{-1/4} \exp(-\frac{1}{2}z^2)\ \Theta[\frac{\pi m}{d} - zi(\frac{\pi}{2d})^{1/2}\ ; \frac{i}{d}] \tag{3.47}$$

3.5.1 Displacements operators of analytic representations

We will express the displacement operators $\mathfrak{Z}$ and $\mathfrak{X}$ in the context of analytic representations. We can write the Eq (3.10) as follows:

$$\mathfrak{X}G(z) = G\left[z - (\frac{2\pi}{d})^{1/2}\right]$$
$$\mathfrak{Z}G(z) = G\left[z + i(\frac{2\pi}{d})^{1/2}\right]\exp\left[(\frac{2\pi}{d})^{1/2} - \frac{\pi}{d}\right]. \tag{3.48}$$

Therefore $\mathfrak{Z}$ and $\mathfrak{X}$ are given by :

$$\mathfrak{X} = \exp\left[-(\frac{2\pi}{d})^{1/2}\ \mathfrak{d}_z\right]$$
$$\mathfrak{Z} = \exp\left[(\frac{2\pi}{d})^{1/2} - \frac{\pi}{d}\right]\exp\left[\ i(\frac{2\pi}{d})^{1/2}\ \mathfrak{d}_z\right]. \tag{3.49}$$

Using Eq(3.49) we can writ the general displacement operators as follows:

$$\mathcal{D}(\alpha,\beta) = \omega(-2^{-1/2}\alpha\beta)\exp\left[\ i\alpha z(\frac{2\pi}{d})^{1/2} - \frac{\alpha^2\pi}{d}\right]$$
$$\times\exp\left[(i\alpha-\beta)(\frac{2\pi}{d})^{1/2}\mathfrak{d}_z\right] \tag{3.50}$$

where α, β are integers in $\mathbb{Z}_d$. Acting out this operator with the function [51] $G(z)$ in Eq (3.31) we get:

$$\mathcal{D}(\alpha,\beta)G(z) = \pi^{-1/4}\exp\left[i\alpha z(\frac{2\pi}{d})^{1/2} - \frac{\alpha^2\pi}{d} - \frac{2^{1/2}i\pi\alpha\beta}{d}\right]$$
$$\times\sum_{m=0}^{d-1}\mathcal{G}_m\ \Theta_3\left[\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}} - \frac{i\pi\alpha}{d} + \frac{\pi\beta}{d};\frac{i}{d}\right] \tag{3.51}$$

3.6 Conclusion

In this chapter, we presented the basic concepts of quantum systems with variables in d-dimensional $\mathbb{Z}_d$. We stated the position and momentum states and Fourier transform. We considered displacement operators in section 3.3, and introduced the Wigner and Wyel functions, while also discussing their properties. In the final section we considered the analytic representation of finite quantum systems.

Chapter 4

Paths of zeros of analytic functions describing finite quantum systems

4.1 Introduction

In this chapter we study the analytic functions on a torus, using Theta functions. It has been shown that these functions have exactly d zeros, which define uniquely the state of the system. We show that as the system evolves in time, the zeros follow d paths on the torus. We study in detail the d paths of the zeros of periodic systems. Each path is characterized by the multiplicity M ,which we back up with $(M = 1, 2, 3, 4)$ as examples , and by a pair of winding numbers (w_1, w_2).In this chapter we also show how two paths with multiplicity $M = 1$, can join into one path with multiplicity $M = 2$.

Displacement operators $\mathfrak{Z}^{\alpha}\mathfrak{X}^{\beta}$ are defined in finite quantum systems.

The interesting part of this chapter is the study these operators to the real power t, where we can see that the paths of the zeros are identical, but they are shifted with respect to each other.

4.2 The zeros of analytic function $G(z)$

Let $G(z)$ be an analytic function , and ζ_n be the zeros of this function, such as $G(\zeta_n) = 0$. We consider the integrals :

$$\mathcal{L}_0 = \oint_\Gamma \frac{dz}{2\pi i} \frac{\partial_z G(z)}{G(z)}; \quad \mathcal{L}_1 = \oint_\Gamma \frac{dz}{2\pi i} \frac{\partial_z G(z)}{G(z)} z. \tag{4.1}$$

Where the first integral inside the contour Γ equal to the number of the zeros of the function $G(z)$ and the second integral gives the sum of these zeros. The analytic function $G(z)$ and Γ uses the quasi-periodicity to perceive that the analytic function $G(z)$ has exactly d zeros, within each cell. using the quasi-periodicity relations of Eq.(3.28) we can prove that :

$$\frac{1}{2\pi i} \oint_{\mathbb{S}} dz \frac{\partial_z G(z)}{G(z)} z = \sum_{n=0}^{d-1} \zeta_n. \tag{4.2}$$

The sum of the zeros ζ_n are given as.

$$\sum_{n=0}^{d-1} \zeta_n = (2\pi)^{\frac{1}{2}} d^{\frac{3}{2}} (M + iN) + d^{3/2} \sqrt{\frac{\pi}{2}} (1 + i). \tag{4.3}$$

Here (M, N) define the cell S where S is the cell $[0, \sqrt{2\pi d}] \times [0, \sqrt{2\pi d}]$.

In infinite systems the zeros do not define uniquely the state. If the $d - 1$

zeros ζ_n are given, the last one can be found from Eq.(4.3), and the function $G(z)$ is given by

$$
\begin{aligned}
G(z) &= \mathcal{N}(\{\zeta_n\}) \\
&\times \exp\left[-i\sqrt{\frac{2\pi}{d}}Nz\right]\prod_{n=1}^{d}\Theta_3\left[\sqrt{\frac{\pi}{2d}}(z-\zeta_n)+\frac{\pi(1+i)}{2};\ i\right]
\end{aligned}
\tag{4.4}
$$

Here N is the integer that labels the cell (as in Eq.(3.43), and $\mathcal{N}(\{\zeta_n\})$ is a normalization constant that dose not depend on z (see section 7 in [51]). In this case we choose the cell with $M = N = 0$.

4.2.1 The analytic representation for a given set of zeros

In this section we assume that d zeros ζ_n in the cell S are given and we will construct the analytic function $G(z)$. In other words, we note that some of the zeros might be equal to each other . We first consider the function:

$$\Psi(z) = \prod_{n=1}^{d}\Theta_3\left[\sqrt{\frac{\pi}{2d}}(z-\zeta_n)+\frac{\pi(1+i)}{2};\ i\right]. \tag{4.5}$$

It is easily shown that $\Psi(z)$ has the given zeros. And the entire function $G(z)/\Psi(z)$ has no zeros and therefore the exponential of this function is:

$$G(z) = \Psi(z)\exp(\phi(z)). \tag{4.6}$$

By taking into consideration the periodicity constraints of Eq (3.40) we can find that :

$$\phi\left[z+(2\pi d)^{1/2}\right]=\phi(z)+2i\pi\mathcal{L};$$
$$\phi\left[z+i(2\pi d)^{1/2}\right]=\phi(z)+2\pi(N+i\mathcal{M}); \tag{4.7}$$

Here $\mathcal{L};\mathcal{M}$ are arbitrary integers and N is the integer entering the constraint of Eq (4.3). We have expended earlier that the growth of the analytic function $G(z)$ is the order 2. So the order of $\Psi(z)$ is 2; therefore, the $\phi(z)$ is a polynomial of maximum possible degree 2. From Eq (4.7), we find that:

$$\phi(z)=-\left(\frac{2\pi}{d}\right)^{1/2}Nzi+C; \tag{4.8}$$

Here C is constant. By inserting Eq(4.8) into (4.6) we find:

$$\begin{aligned} G(z) &= \Psi(z)\exp\ (C-\left(\frac{2\pi}{d}\right)^{1/2}Nzi) \\ &= \exp(C)\exp(-i\left(\frac{2\pi}{d}\right)^{1/2}Nz)\prod_{n=1}^{d}\Theta_3[\omega_n(z);i]\ ; \\ \omega_n(z) &= \left[\sqrt{\frac{\pi}{2d}}(z-\zeta_n)+\frac{\pi(1+i)}{2}\right] \end{aligned} \tag{4.9}$$

Where $N\in\mathbb{Z}$ and $\exp(C)=A$ is normalization constant. To calculate the coefficient $\mathcal{G}_m$ we insert d arbitrary numbers $(z_0,z_1,....,z_{d-1})$ and solve the system with d equations and d unknowns. We take the normalisation coefficients equal to one, and after the calculation we normalise $\mathcal{G}_m$. This is shown by the equation below:

$$G(z_n) = \pi^{-1/4} \sum_{m=0}^{d-1} \mathcal{G}_m \, \Theta_3 \left[\frac{\pi m}{d} - z_n \sqrt{\frac{\pi}{2d}}; \frac{i}{d} \right]. \tag{4.10}$$

As an example, when $d = 4$ and if we have:

$$\zeta_0 = 1 + 3.02i; \; \zeta_1 = 3.02 + 3i; \; \zeta_2 = 1 + 3i; \tag{4.11}$$

We can get the last zero using Eq.(4.3), so the third zero is equal to $\zeta_3 = 5.01 + 1.9i;$.

We choose four arbitrary values $0, -1, 1, 2$ and insert them with the zeros into Eq.(4.9). We then find $G(0), G(1), G(2), G(3)$ to insert them in Eq (4.10), and we end up with one system with four equations and four unknowns.

In this case we get:

$$\begin{pmatrix} \mathcal{G}_0 \\ \mathcal{G}_1 \\ \mathcal{G}_2 \\ \mathcal{G}_3 \end{pmatrix} = \begin{pmatrix} 0.2297 + 0.1287i \\ 0.1404 + 0.1358i \\ 0.8132 - 0.0300i \\ 0.3520 - 0.3262i \end{pmatrix} \tag{4.12}$$

4.2.2 Paths of zeros and time evolution

Let $\mathcal{H}$ be the Hamiltonian of the system ($d \times d$ Hermitian matrix $\mathcal{H}_{mn}$), and the state:

$$|\mathcal{G}(0)\rangle = \sum_{n=1}^{d-1} \mathcal{G}_m(0)|X; m\rangle; \ t = 0 \tag{4.13}$$

As the system evolves at time t and each zero ζ_n follows a path $\zeta_n(t)$ into:

$$|\mathcal{G}(t)\rangle = \exp(-it\mathcal{H})|\mathcal{G}(0)\rangle = \sum_{n=1}^{d-1} \mathcal{G}_m(t)|X; m\rangle. \tag{4.14}$$

Where $\mathcal{H}$ is the Hermitian

$$G(z;t) = \pi^{-1/4} \sum_{m=0}^{d-1} \mathcal{G}_m(t) \ \Theta_3 \left[\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}}; \frac{i}{d} \right] \tag{4.15}$$

We consider infinitesimal changes to the coefficients from $\mathcal{G}_m$ to $\mathcal{G}_m + \Delta\mathcal{G}_m$, where:

$$\Delta\mathcal{G}_m = i\Delta t \sum_n \mathcal{H}_{mn}\mathcal{G}_n \tag{4.16}$$

Then the zeros will change from ζ_n to $\zeta_n + \Delta\zeta_n$. From Eq.(4.4) and (4.15) we get:

$$\begin{aligned} &\pi^{-1/4} \sum_{m=0}^{d-1} (\mathcal{G}_m + \Delta\mathcal{G}_m) \ \Theta_3 \left[\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}}; \frac{i}{d} \right] \\ &= \mathcal{N}(\{\zeta_k\}) \prod_{n=1}^{d} \Theta_3 \left[\sqrt{\frac{\pi}{2d}}(z - \zeta_n - \Delta\zeta_n) + \frac{\pi(1+i)}{2}; \ i \right]. \end{aligned} \tag{4.17}$$

Using Taylor expansion on the right hand side, we get the following:

$$\begin{aligned}
&\pi^{-1/4}\sum_{m=0}^{d-1}\Delta\mathcal{G}_m\ \Theta_3\left[\frac{\pi m}{d}-z\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right]\\
&=-\mathcal{N}(\{\zeta_k\})\sqrt{\frac{\pi}{2d}}\sum_{j=1}^{d}A_j(z)\Theta_3'\left[\sqrt{\frac{\pi}{2d}}(z-\zeta_j)+\frac{\pi(1+i)}{2};\ i\right]\Delta\zeta_j\\
&A_j(z)=\prod_{m\neq j}\Theta_3\left[\sqrt{\frac{\pi}{2d}}(z-\zeta_m)+\frac{\pi(1+i)}{2};\ i\right].
\end{aligned}\tag{4.18}$$

We insert $z=\zeta_n$ on both sides of this equation. For $j\neq n$, we get $A_j(\zeta_n)=0$, then we can see that:

$$\begin{aligned}
&\pi^{-1/4}\sum_{m=0}^{d-1}\Delta\mathcal{G}_m\ \Theta_3\left[\frac{\pi m}{d}-\zeta_n\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right]\\
&=-\mathcal{N}(\{\zeta_k\})\sqrt{\frac{\pi}{2d}}A_n(\zeta_n)\Theta_3'\left[\frac{\pi(1+i)}{2};\ i\right]\Delta\zeta_n\\
&A_n(\zeta_n)=\prod_{m\neq n}\Theta_3\left[\sqrt{\frac{\pi}{2d}}(\zeta_n-\zeta_m)+\frac{\pi(1+i)}{2};\ i\right].
\end{aligned}\tag{4.19}$$

Using the derivative of theta function

$$\Theta_3'(u,\tau)=\frac{d\Theta_3}{du}=i\sum_{-\infty}^{\infty}2n\exp(i\pi\tau n^2+2inu).\tag{4.20}$$

We found numerically that:

$$\Theta_3'\left[\frac{\pi(1+i)}{2};\ i\right]=1.9888i.\tag{4.21}$$

Therefore, we have analytical expressions for the derivatives of the functions $\zeta_n(\mathcal{G}_0, ..., \mathcal{G}_{d-1})$:

$$\frac{\partial \zeta_n}{\partial \mathcal{G}_m} = -\frac{\pi^{-1/4}\Theta_3\left[\frac{\pi m}{d} - \zeta_n\sqrt{\frac{\pi}{2d}};\, \frac{i}{d}\right]}{\mathcal{N}(\{\zeta_k\})\sqrt{\frac{\pi}{2d}}A_n(\zeta_n)\Theta_3'\left[\frac{\pi(1+i)}{2};\; i\right]} \tag{4.22}$$

We use numerical calculations for them as:

$$\zeta_n + \Delta\zeta_n = \zeta_n + \sum_m \frac{\partial \zeta_n}{\partial \mathcal{G}_m}\Delta\mathcal{G}_m = \zeta_n + i\Delta t\sum_{m,k}\frac{\partial \zeta_n}{\partial \mathcal{G}_m}\mathcal{H}_{mk}\mathcal{G}_k \tag{4.23}$$

So in each step of the iteration process $\mathcal{N}(\{\zeta_k\})$ is calculated as:

$$\mathcal{N}(\{\zeta_k\}) = \frac{\pi^{-1/4}\sum_{m=0}^{d-1}\mathcal{G}_m\,\Theta_3\left[\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}};\, \frac{i}{d}\right]}{\prod_{n=1}^{d}\Theta_3\left[\sqrt{\frac{\pi}{2d}}(z-\zeta_n) + \frac{\pi(1+i)}{2};\; i\right]} \tag{4.24}$$

The $\mathcal{N}(\{\zeta_k\})$ dose not depend on z and any value of z can be used for its numerical calculation.

Since $\sum_m |\mathcal{G}_m|^2 = 1$ the $\Delta\mathcal{G}_m$ are subject to the constraint.

$$\sum_m [\mathcal{G}_m^*\Delta\mathcal{G}_m + \mathcal{G}_m(\Delta\mathcal{G}_m)^*] = 0 \tag{4.25}$$

4.3 Periodic systems of the zeros

In this section we consider Hamiltonians with periodic systems such that $\exp(iTH) = \mathbf{1}$ for some T. This occurs when the ratios of the eigenvalues of H are rational numbers. That mean, when the eigenvalues of H are denoted by K_i (where $i = 0, 1, 2, ..., d-1$) and $\frac{K_1}{K_i}$ is a rational number, then the system is periodic.In this case the d paths of the zeros are in general closed curves on the torus.

Results comparable to those in section $(4.3.1 - 4.3.3)$ have been presented in [52],[61], using a purely numerical method.

4.3.1 Multiplicity M of paths of zeros :

The d paths of the zeros in general are closed curves on the torus. In some cases M of the zeros follow the same path. We call that, this path has multiplicity M.

Paths with multiplicity M=1:

In this case we consider the Hamiltonian when $d = 3$

$$H = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{4.26}$$

The eigenvalues of this Hamiltonian are $-2, 2, 2$. To calculate the paths of the zeros we use Eq. 4.23. We assume that the paths of the zeros are denoted by $\zeta_0(t), \zeta_1(t), \zeta_2(t)$ where $t = 0$ the zeros of the analytic function are

$\zeta_0(0), \zeta_1(0), \zeta_2(0)$ the zeros are the following:

$$\begin{aligned} &\zeta_0(0) = 1.25 + 2i; \qquad \zeta_1(0) = 2.59 + 2.56i; \\ &\zeta_2(0) = 2.66 + 1.95i; \end{aligned} \tag{4.27}$$

In this case we have three paths with multiplicity $M = 1$ that mean all of the zeros follow their own path. After a period $T = \pi$, the zeros return to their initial position. For clarity, the figures show regions which might be larger or smaller than one cell(which is a square with each side equal to $(\sqrt{2\pi d})$. The results are shown in Fig (4.1). We also note that:

$$\begin{aligned} \zeta_0(T) = \zeta_0(0); \quad \zeta_1(T) = \zeta_1(0); \\ \zeta_2(T) = \zeta_2(0). \end{aligned} \tag{4.28}$$

We consider the Hamiltonian when $d = 5$

$$H = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}. \tag{4.29}$$

The eigenvalues of this Hamiltonian are $-0.5, 0.5, 0.5, 0.5, 0.5$. We assume that at $t = 0$ the zeros here are

$$\begin{aligned} &\zeta_0(0) = 1.3 + 4.13i; \qquad \zeta_1(0) = 5.72 + 2.03i; \\ &\zeta_2(0) = 0.11 + 1.62i; \qquad \zeta_3(0) = 3 + 5.4i. \\ &\zeta_4(0) = 3.89 + 0.81i \quad . \end{aligned} \tag{4.30}$$

In this case we have five paths with multiplicity $M = 1$ that means all the zeros follow their own path. After a period $T = 4\pi$, the zeros return to their initial position. The results are shown in Fig (4.2).

$$\begin{gathered} \zeta_0(T) = \zeta_0(0); \quad \zeta_1(T) = \zeta_1(0); \\ \zeta_2(T) = \zeta_2(0); \quad \zeta_3(T) = \zeta_3(0); \quad \zeta_4(T) = \zeta_4(0). \end{gathered} \tag{4.31}$$

Paths with multiplicity M=2

In this section, we show that the perturbation in the initial values of the zeros divide the path with multiplicity $M = 2$ into two paths. We consider the Hamiltonian when $d = 3$

$$H = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{4.32}$$

The eigenvalues of this Hamiltonian are $-2, 2, 2$ with the period $T = 2\pi$, the zeros are as follows:

$$\begin{aligned} &\zeta_0(0) = 1.52 + 2.43i; \quad \zeta_1(0) = 1.45 + 2.22i; \\ &\zeta_2(0) = 3.53 + 1.85i; \end{aligned} \tag{4.33}$$

In this case, we get:

$$\begin{aligned} \zeta_0(T) = \zeta_1(0); \quad \zeta_1(T) = \zeta_0(0); \\ \zeta_2(T) = \zeta_2(0). \end{aligned} \tag{4.34}$$

Therefore the first two zeros following the same path with multiplicity $M = 2$. The last zero follows its own path, as shown in Figure(4.3)

We consider the Hamiltonian when $d = 4$

$$H = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}. \tag{4.35}$$

After a period $T = 2\pi$. We assume that at $t = 0$ the zeros are as follows:

$$\begin{aligned} &\zeta_0(0) = 1 + 3.02i; \quad \zeta_1(0) = 3.02 + 3i; \\ &\zeta_2(0) = 1 + 3i; \qquad \zeta_3(0) = 5 + 1i; \end{aligned} \tag{4.36}$$

The results are shown in Fig(4.4). In this case we have two paths with multiplicity $M = 1$ and another path with multiplicity $M = 2$. Here

$$\zeta_0(T) = \zeta_2(0); \quad \zeta_2(T) = \zeta_0(0);$$
$$\zeta_1(T) = \zeta_1(0); \quad \zeta_3(T) = \zeta_3(0). \tag{4.37}$$

We also consider the Hamiltonian of Eq(4.29). After a period $T = 2\pi$ and assuming that $t = 0$, the following zeros are derived:

$$\zeta_0(0) = 0.3 + 3.66i; \qquad \zeta_1(0) = 3.72 + 4.03i;$$
$$\zeta_2(0) = 1.61 + 2.72i; \qquad \zeta_3(0) = 3.5 + 2.9i.$$
$$\zeta_4(0) = 4.89 + 0.81i. \tag{4.38}$$

The results are shown in Fig(4.5). In this case we have three paths with multiplicity $M = 1$ and another path with multiplicity $M = 2$.

$$\zeta_0(T) = \zeta_2(0); \quad \zeta_2(T) = \zeta_0(0);$$
$$\zeta_1(T) = \zeta_1(0); \quad \zeta_3(T) = \zeta_3(0); \quad \zeta_4(T) = \zeta_4(0). \tag{4.39}$$

Paths with multiplicity M=3

In this case we consider the Hamiltonian of Eq.(4.26) and assume that at $t = 0$ the zeros are as follows:

$$\zeta_0(0) = 1.56 + 2.49i; \qquad \zeta_1(0) = 1.99 + 2.16i;$$
$$\zeta_2(0) = -1.39 + 1.85i; \tag{4.40}$$

The results are shown in Fig(4.6). We can see that we have one path with multiplicity $M = 3$ which means all zeros following the same path are:

$$\zeta_0(T) = \zeta_1(0); \quad \zeta_1(T) = \zeta_2(0);$$
$$\zeta_2(T) = \zeta_0(0). \tag{4.41}$$

We also consider the Hamiltonian of Eq.(4.35) and assume that at $t = 0$ the zeros are:

$$\zeta_0(0) = 1 + 2.02i; \quad \zeta_1(0) = 4.02 + 3.5i;$$
$$\zeta_2(0) = 1 + 1.5i; \quad \zeta_3(0) = 4 + 3i; \tag{4.42}$$

The results are shown in Fig(4.7). In this case we have one path with multiplicity $M = 1$ and another path with multiplicity $M = 3$, here.

$$\zeta_0(T) = \zeta_3(0); \quad \zeta_3(T) = \zeta_1(0);$$
$$\zeta_1(T) = \zeta_0(0); \quad \zeta_2(T) = \zeta_2(0). \tag{4.43}$$

Paths with multiplicity M=4

We consider the Hamiltonian when $d = 4$

$$H = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}. \tag{4.44}$$

After a period $T = 2\pi$. We assume that at $t = 0$ the zeros are the following:

$$\begin{aligned} &\zeta_0(0) = 2 + 2.02i; \quad \zeta_1(0) = 2.02 + 3i; \\ &\zeta_2(0) = 1 + 4i; \qquad \zeta_3(0) = -0.01 + i; \end{aligned} \tag{4.45}$$

The results are shown in Fig(4.8). In this case all zeros are following the same path $M = 4$.

$$\begin{aligned} \zeta_0(T) = \zeta_3(0); \quad \zeta_3(T) = \zeta_2(0); \\ \zeta_2(T) = \zeta_1(0); \quad \zeta_1(T) = \zeta_0(0). \end{aligned} \tag{4.46}$$

Now we consider the Hamiltonian in Eq(4.29) when $d = 5$. We can see that after a period $T = 2\pi$. The zeros are the following:

$$\begin{aligned}
&\zeta_0(0) = 0.69 + 1.57i; \qquad &&\zeta_1(0) = 3.42 + 5.03i;\\
&\zeta_2(0) = 2.31 + 1.62; &&\zeta_3(0) = 2.4 + 3.8i.\\
&\zeta_4(0) = -0.41 + 1.99i && \qquad\qquad\qquad\qquad (4.47)
\end{aligned}$$

The results are shown in Fig(4.9). In this case we have one path with multiplicity $M = 1$ and another path with multiplicity $M = 4$

$$\begin{aligned}
\zeta_0(T) = \zeta_2(0); \quad \zeta_2(T) = \zeta_0(0);\\
\zeta_1(T) = \zeta_1(0); \quad \zeta_3(T) = \zeta_3(0). \qquad (4.48)
\end{aligned}$$

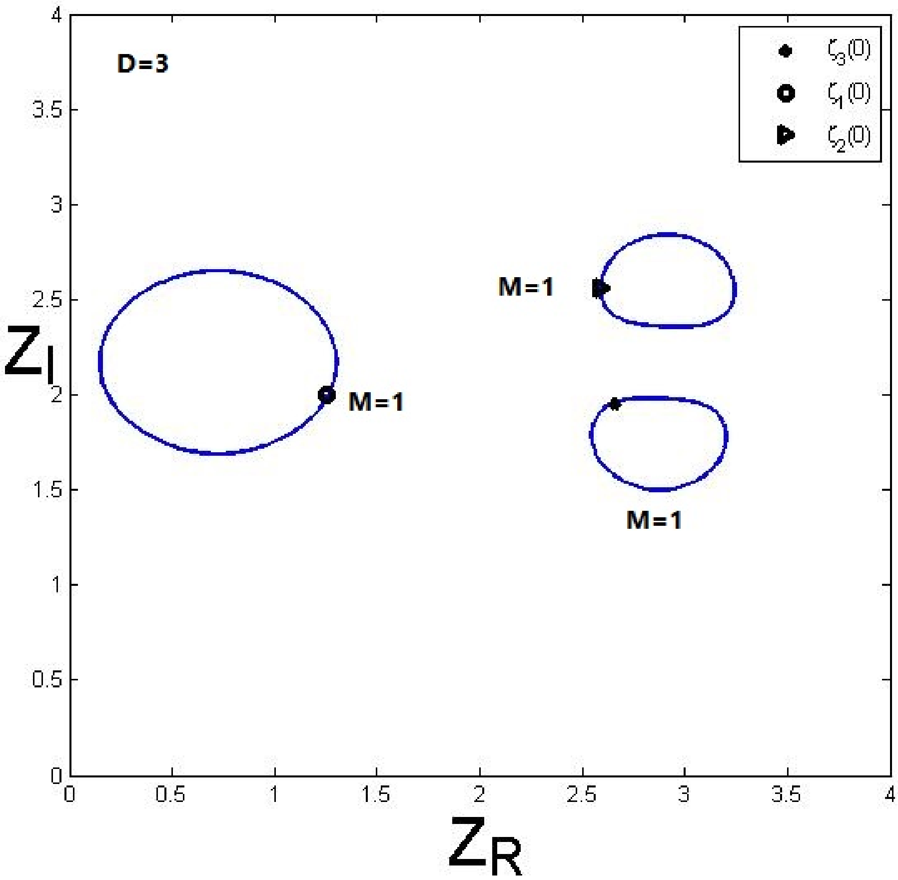

Figure 4.1: Paths of the zeros for the Hamiltonian of Eq.(4.26). The period is $T = 2\pi$. At $t = 0$ the zeros are given in Eq.(4.66). The cell is a square with each side equal to 4.3

.

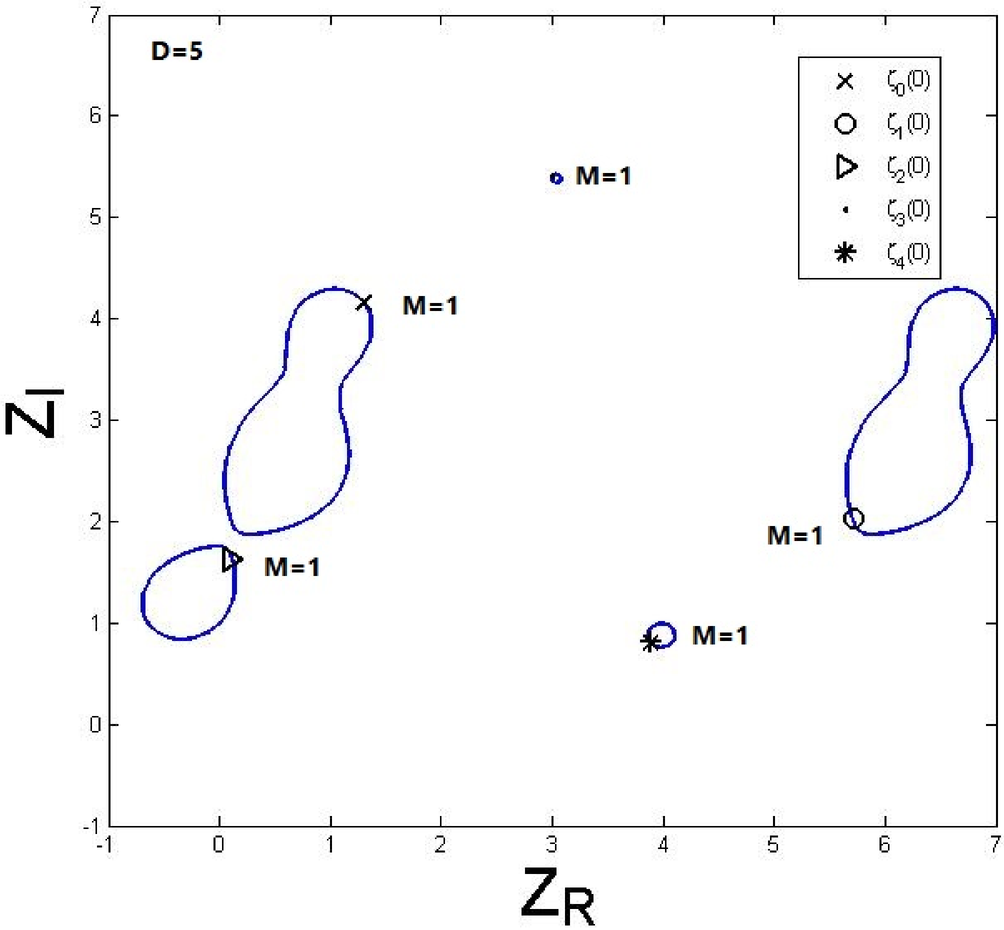

Figure 4.2: Paths of the zeros for the Hamiltonian of Eq.(4.29). The period is $T = 2\pi$. At $t = 0$ the zeros are given in Eq.(4.70). The cell is a square with each side equal to 5.6

.

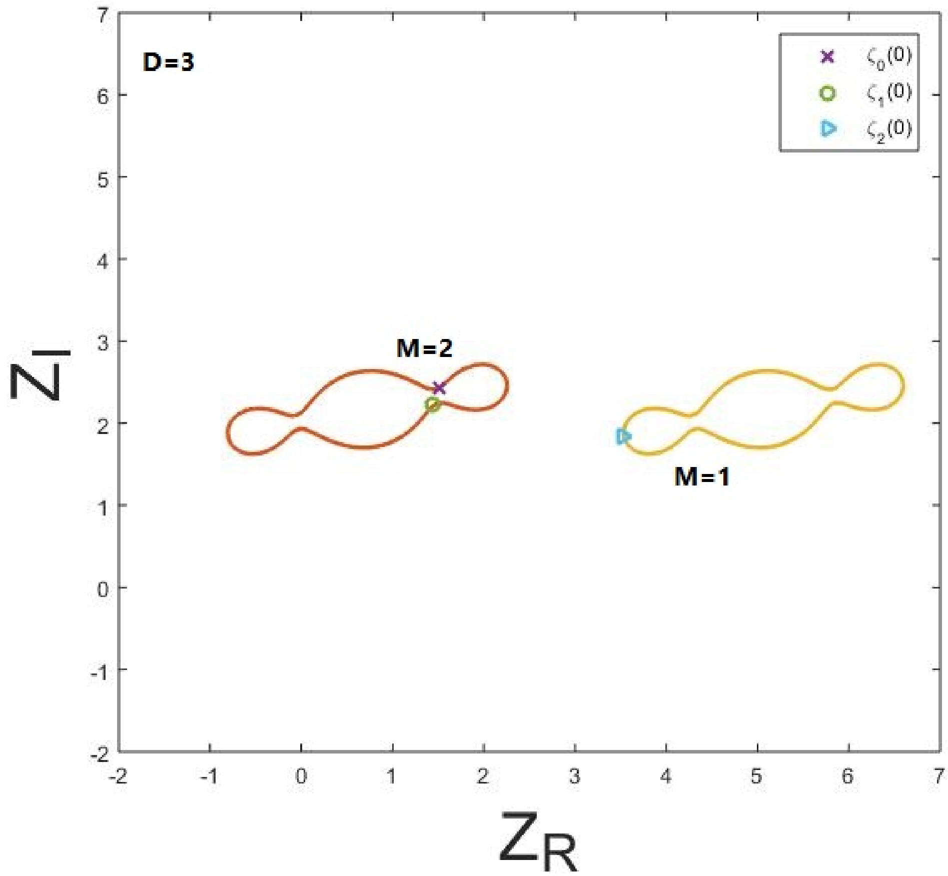

Figure 4.3: Paths of the zeros for the Hamiltonian of Eq.(4.32). The period is $T = 2\pi$. At $t = 0$ the zeros are given in Eq.(4.33). The cell is a square with each side equal to 4.3.

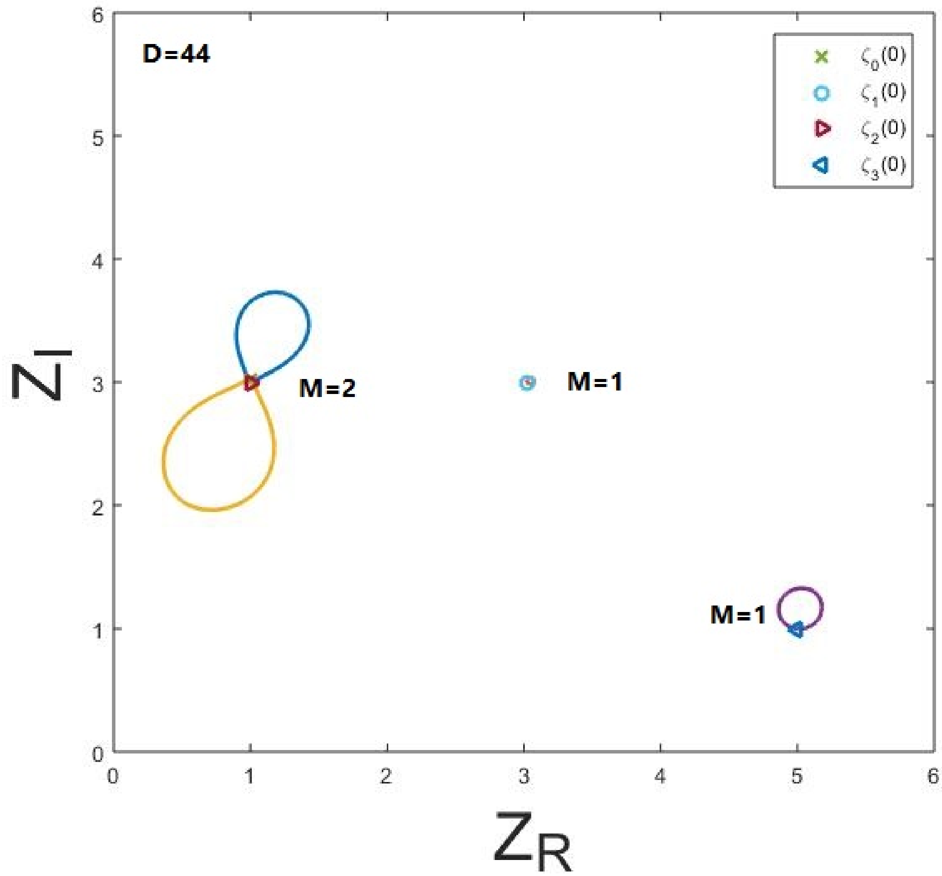

Figure 4.4: Paths of the zeros for the Hamiltonian of Eq(4.35). The period is $T = 2\pi$. At $t = 0$ the zeros are given in Eq(4.45). The cell is a square with each side equal to 5.1.

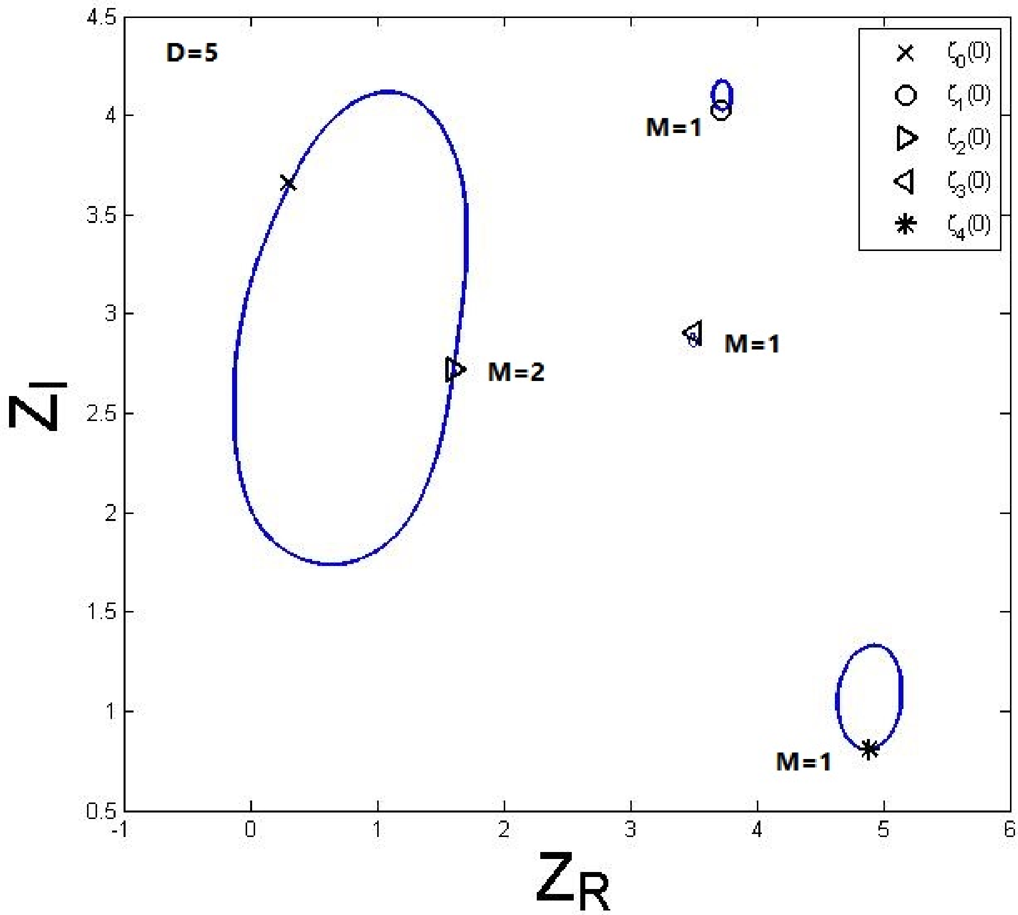

Figure 4.5: Paths of the zeros for the Hamiltonian of Eq(4.29). The period is $T = 2\pi$. At $t = 0$ the zeros are given in Eq(4.38). The cell is a square with each side equal to 5.6.

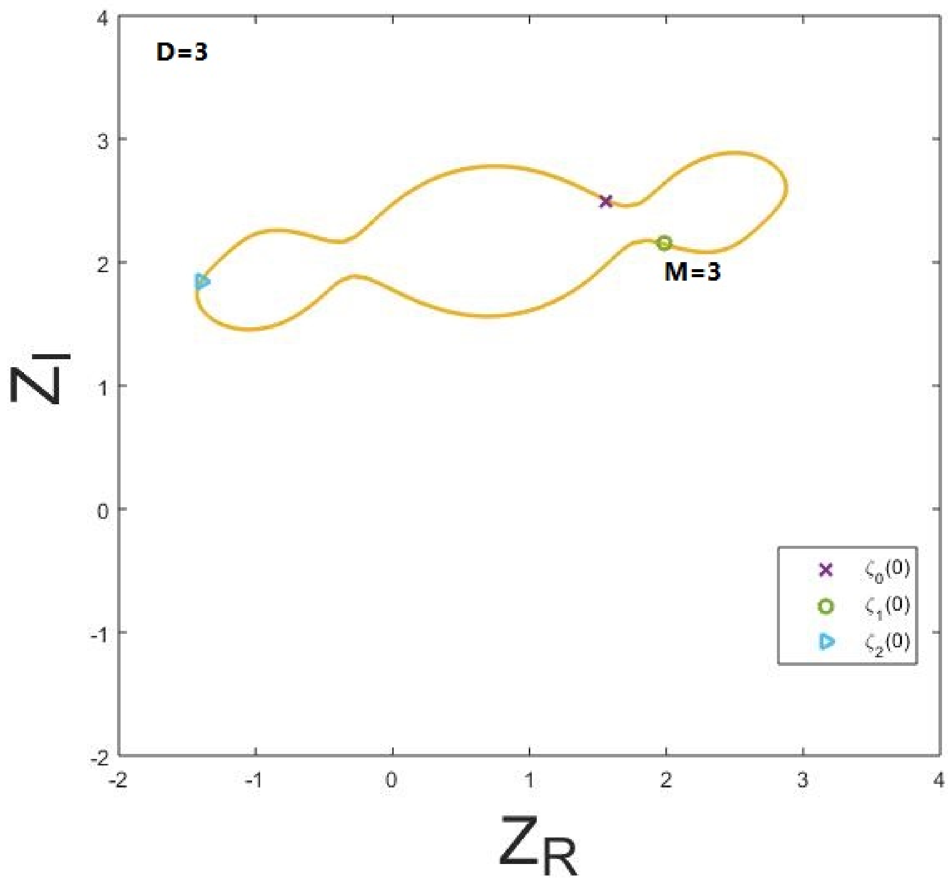

Figure 4.6: Paths of the zeros for the Hamiltonian of Eq.(4.26). The period is $T = 2\pi$. At $t = 0$ the zeros are given in Eq.(4.50). The cell is a square with each side equal to 4.3.

.

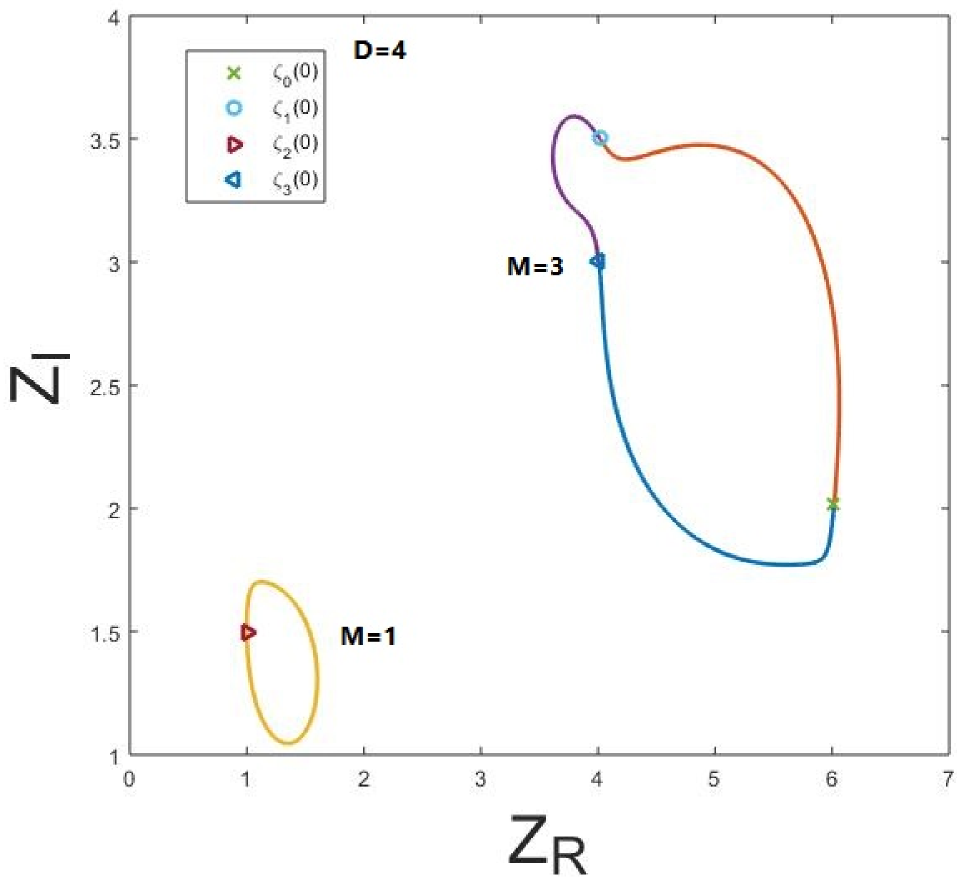

Figure 4.7: Paths of the zeros for the Hamiltonian of Eq.(4.35).The period is $T = 2\pi$. At $t = 0$ the zeros are given in Eq.(4.42). The cell is a square with each side equal to 5.1.

.

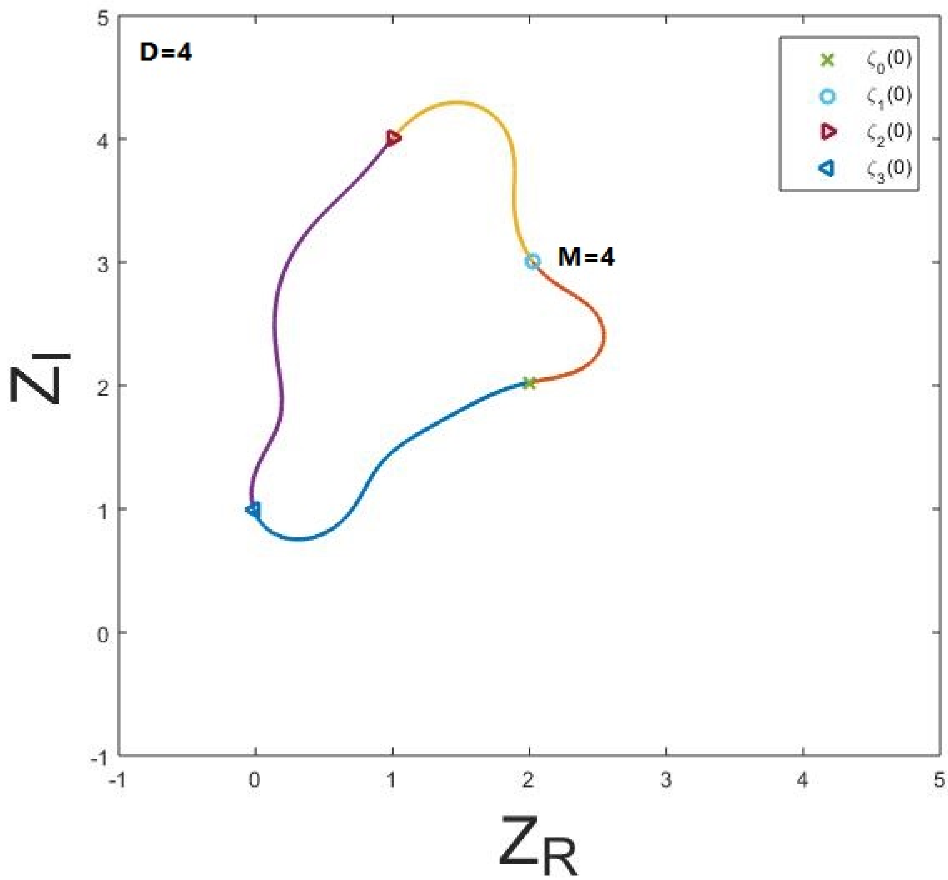

Figure 4.8: Paths of the zeros for the Hamiltonian of Eq.(4.44). The period is $T = 2\pi$. At $t = 0$ the zeros are given in Eq.(4.45). The cell is a square with each side equal to 5.1.

.

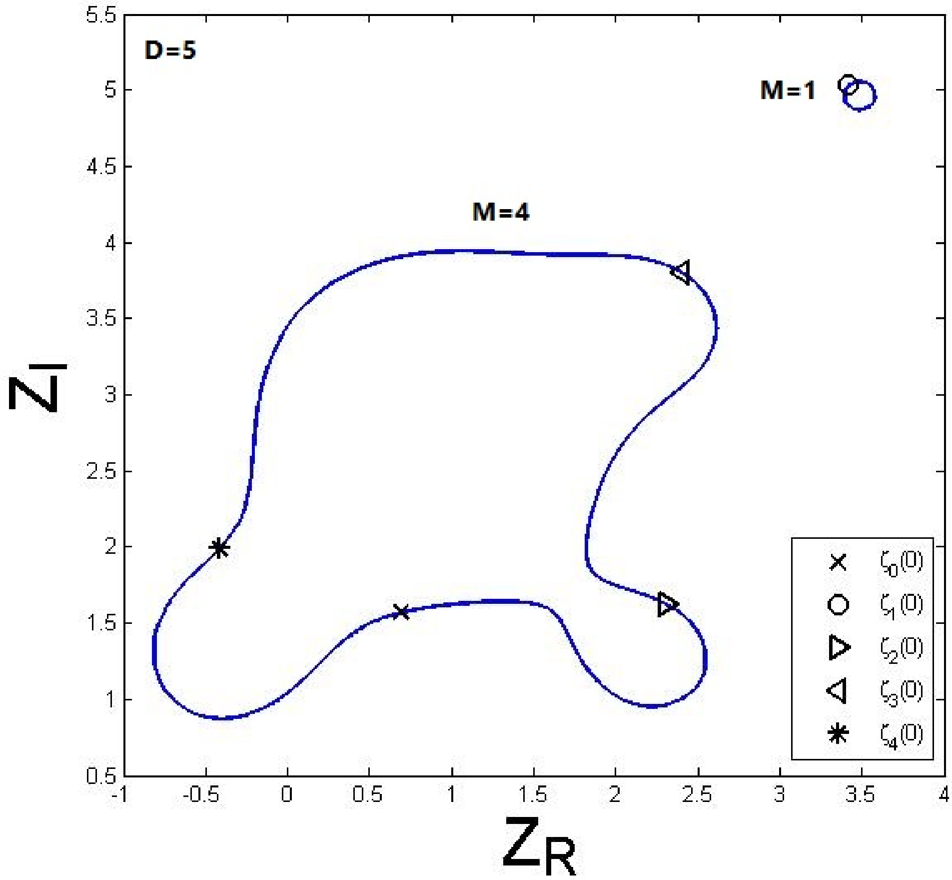

Figure 4.9: Paths of the zeros for the Hamiltonian of Eq(4.29).The period is $T = 2\pi$. At $t = 0$ the zeros are given in Eq.(4.47). The cell is a square with each side equal to 5.6.

.

4.3.2 Winding numbers (w_1, w_2) of paths of zeros :

The winding number of a closed curve in the plane around a given point is an integer representing the total number of times that curve travels counter clockwise around the point. The winding number depends on the orientation of the curve, and is negative if the curve travels around the point clockwise. In this case we consider the Hamiltonian when $d = 5$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.49}$$

In this case the period is $T = 2\pi$. And at $t = 0$ the zeros are the following:

$$\begin{aligned} &\zeta_0(0) = 0.69 + 1.57i; && \zeta_1(0) = 3.42 + 5.03i; \\ &\zeta_2(0) = 2.31 + 1.62; && \zeta_3(0) = 2.4 + 3.8i. \\ &\zeta_4(0) = -0.41 + 1.99i \end{aligned} \tag{4.50}$$

The paths of zeros are shown in Fig.4.10. The winding numbers (w_1, w_2) of the five paths are $(0, 1)$,$(0, 0)$,$(0, 0)$,$(0, 0)$,$(0, 0)$,.

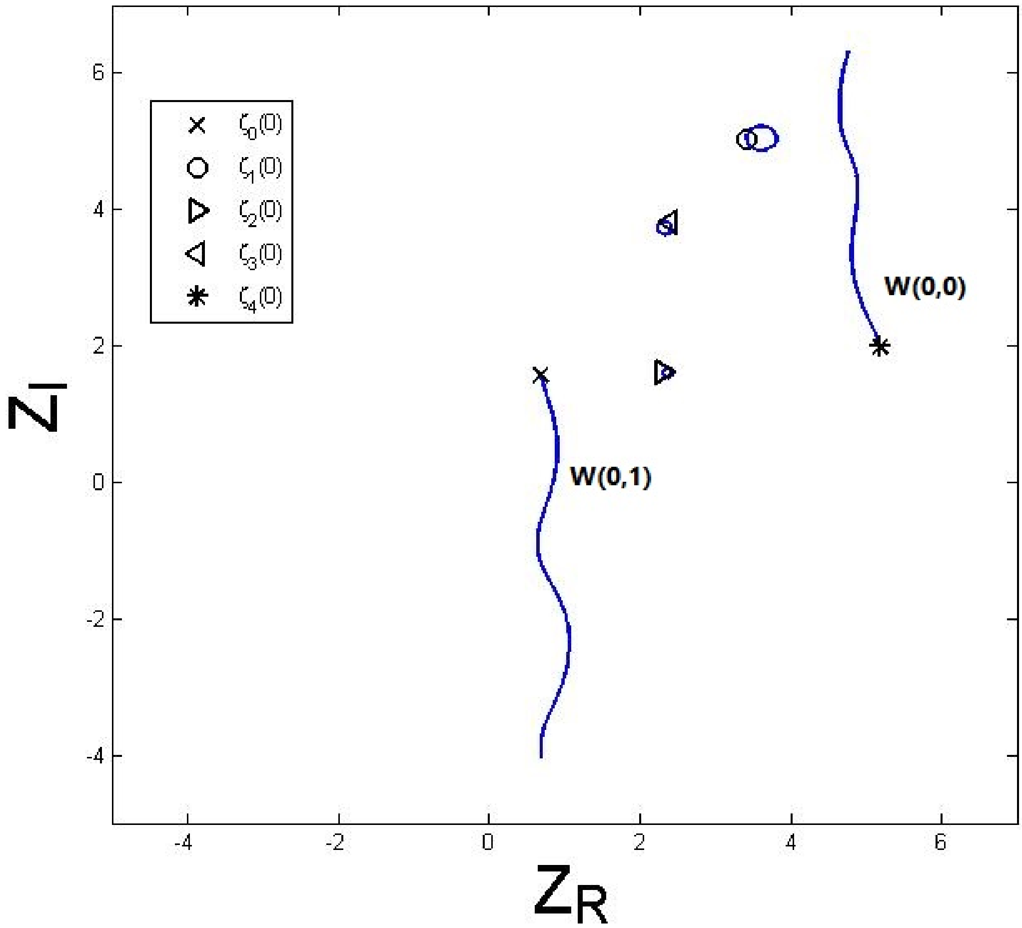

Figure 4.10: Paths of the zeros for the Hamiltonian of Eq.(4.49). At $t = 0$ the zeros are given in Eq.(4.50) The cell is a square with each side equal to 5.6

4.3.3 Joining of two paths of zeros into a single path

We consider the Hamiltonian:

$$H = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}. \tag{4.51}$$

In this case the period is $T = 4\pi$. We consider two cases where at $t = 0$ the zeros are given by

$$\begin{aligned} &\zeta_0(0) = 1.3 + 4.16i; \quad \zeta_1(0) = 5.72 + 2.03i; \\ &\zeta_2(0) = 0.11 + 1.62i; \quad \zeta_3(0) = 3 + 5.4i; \\ &\zeta_4(0) = 3.89 + 0.81i. \end{aligned} \tag{4.52}$$

And also by:-

$$\begin{aligned} &\zeta_0(0) = 1.3 + 4.16i; \quad \zeta_1(0) = 5.72 + 2.03i; \\ &\zeta_2(0) = 0.11 + 1.82i; \quad \zeta_3(0) = 3 + 5.2i; \\ &\zeta_4(0) = 3.89 + 0.81i. \end{aligned} \tag{4.53}$$

The paths of zeros in these two examples are shown in Figs.4.11,and 4.12, correspondingly. In these figures we see how by changing the initial zeros slightly, two paths (highlighted)with multiplicity 1, join together into one path (highlighted) with multiplicity 2.

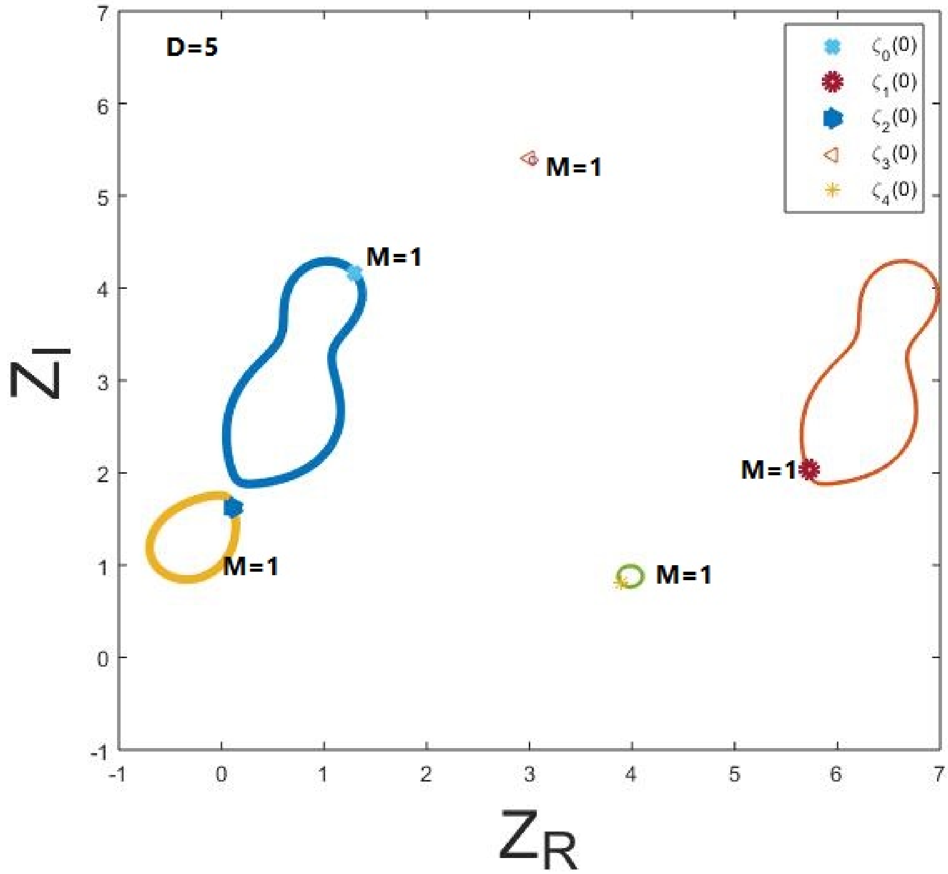

Figure 4.11: Paths of the zeros for the Hamiltonian of Eq(4.51). At $t = 0$ the zeros are given in Eq(4.52) The cell is a square with each side equal to 5.6

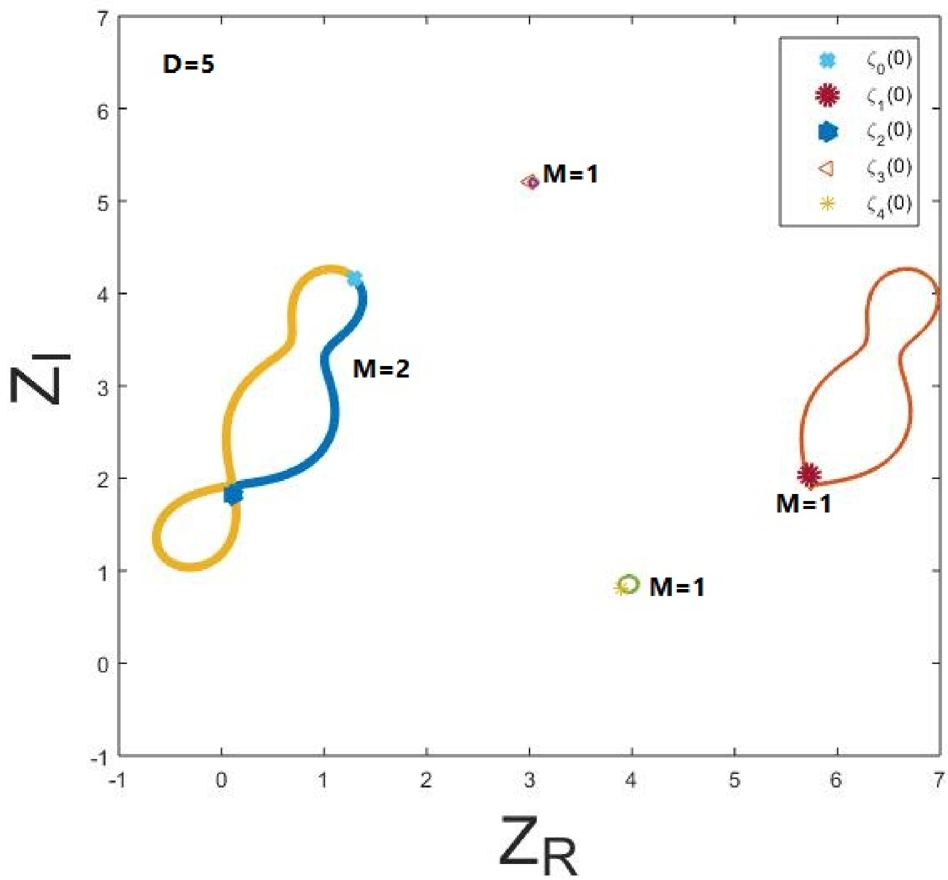

Figure 4.12: Paths of the zeros for the Hamiltonian of Eq(4.51). At $t = 0$ the zeros are given in Eq(4.53) The cell is a square with each side equal to 5.6

4.3.4 Zeros of the analytic representation of $\mathfrak{X}^t|g\rangle$

We define the displacement operators in the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space as:

$$
\begin{aligned}
&\mathfrak{Z} = \sum_m \omega(m)|X;m\rangle\langle X;m| = \sum_m |P;m+1\rangle\langle P;m| \\
&\mathfrak{X} = \sum_m \omega(-m)|P;m\rangle\langle P;m| = \sum_m |X;m+1\rangle\langle X;m| \\
&\mathfrak{X}^d = \mathfrak{Z}^d = \mathbf{1}; \qquad \mathfrak{X}^\beta \mathfrak{Z}^\alpha = \mathfrak{Z}^\alpha \mathfrak{X}^\beta \omega(-\alpha\beta).
\end{aligned} \tag{4.54}
$$

Here $\alpha, \beta \in \mathbb{Z}(d)$. In this section we study the zeros of the analytic representation of $\mathfrak{X}^t|\mathcal{G}\rangle$ which we denote as $G(z;t)$, where $t \in \mathbb{R}$. The operator $\mathfrak{X}^t$ can be viewed as a time evolution operator $\exp(itH)$ with Hamiltonian $H = -i \ln \mathfrak{X}$ (the logarithm is multi-valued and we take the principal value).
Let $|\mathcal{G}\rangle = \sum_m \widetilde{\mathcal{G}}_m |P;m\rangle$,where $\widetilde{\mathcal{G}}_m$ are the Fourier transforms of the $\mathcal{G}_m$ in Eq (3.29). The state $\mathfrak{X}^t|\mathcal{G}\rangle = \sum [\omega(-m)]^t \widetilde{\mathcal{G}}_m |P;m\rangle$ is represented by the function:

$$
\begin{aligned}
\mathcal{G}(z;t) = \pi^{-1/4} \exp\left(\frac{z^2}{2}\right) \sum_{m=0}^{d-1} \exp(-\frac{2i\pi mt}{d}) \widehat{\mathcal{G}}_m \\
\times \Theta_3 \left[\frac{\pi m}{d} - iz\sqrt{\frac{\pi}{2d}}; \frac{i}{d}\right].
\end{aligned} \tag{4.55}
$$

Here we used the fact that the momentum $|P,m\rangle$ is represented by the function [57]:

$$
\pi^{-1/4} \exp\left(\frac{z^2}{2}\right) \Theta_3 \left[\frac{\pi m}{d} - iz\sqrt{\frac{\pi}{2d}}; \frac{i}{d}\right]. \tag{4.56}
$$

Let $\zeta_n(t)$ where $n = 0, ..., d-1$ be the zeros of $G(z;t)$, which mean that:

$$G[\zeta_n(t);t] = 0 \tag{4.57}$$

Where n labels the different paths of zeros.

Proposition 4.3.1. *Through the Eq*(4.55) *we can seen that each path of the zeros $\zeta_{n+\beta}$ of $G(z,t)$, is a shifted version (in one direction) of another path ζ_n.In this case the position of the zero on each path at a certain time, is the same as the position of the zero on another path, at a different time:*

$$\zeta_{n+\beta}(t+\beta) = \zeta_n(t) + \beta\sqrt{\frac{2\pi}{d}}; \quad n, \beta \in \mathbb{Z}(d). \tag{4.58}$$

In this status we assume $G[\zeta_n(t);t] = 0$ and prove that $G[\zeta_{n+\beta}(t+\beta);t+\beta] = 0$, where the 'new path' $\zeta_{n+\beta}(t+\beta)$ is given in Eq(4.58). Then we express the Theta function as a sum as in Eq (3.32) and change the summation from $n \in \mathbb{Z}$ into $L_n = n - t \in \mathbb{Z} - t$.

To prove that we define the following analytic representation for $\mathfrak{X}^t|p;m\rangle$:

$$G(z;t;\mathcal{G}) = \exp(-\frac{z^2}{2})\sum_{m=0}^{d-1}\langle P;m|\mathfrak{X}^t|\mathcal{G}\rangle\theta_3[\frac{\pi m}{d} - iz\sqrt{\frac{\pi}{2d}};\frac{i}{d}]$$
$$= \pi^{\frac{-1}{4}}\exp(-\frac{z^2}{2})\exp(2tz\sqrt{\frac{\pi}{2d}} - \frac{\pi t^2}{d})$$
$$\sum_{m=0}^{d-1}\widetilde{\mathcal{G}}_m\sum_{L_n=-\infty}^{\infty}\exp(\frac{2i\pi m L_n}{d})\exp(2L_n z\sqrt{\frac{\pi}{2d}} - \frac{\pi L_n^2}{d} - \frac{2\pi L_n t}{d})$$
$$. \qquad (4.59)$$

So if :

$$G(\zeta_n;t;\mathcal{G}) = 0 \qquad (4.60)$$

Then :

$$G(\zeta_n - \beta\sqrt{\frac{2\pi}{d}};t;\mathcal{G}) = 0 \qquad (4.61)$$

proves :

$$G(\zeta_n - \beta\sqrt{\frac{2\pi}{d}};t;\mathcal{G}) = \pi^{\frac{-1}{4}}\exp[-\frac{(\zeta_n - \beta\sqrt{\frac{2\pi}{d}})^2}{2}]\exp[2t(\zeta_n - \beta\sqrt{\frac{2\pi}{d}})\sqrt{\frac{\pi}{2d}} - \frac{\pi t^2}{d}]$$
$$\sum_{m=0}^{d-1}\widetilde{\mathcal{G}}_m\sum_{L_n=-\infty}^{\infty}\exp(\frac{2i\pi m L_n}{d})\exp[2L_n(\zeta_n - \beta\sqrt{\frac{2\pi}{d}})\sqrt{\frac{\pi}{2d}} - \frac{\pi L_n^2}{d} - \frac{2\pi L_n t}{d}]$$
$$(4.62)$$

$$G(\zeta_n - \beta\sqrt{\frac{2\pi}{d}}; t; \mathcal{G}) = \pi^{\frac{-1}{4}} \exp[-\frac{\zeta_n^2}{2} - \beta^2\frac{\pi^2}{d} + \zeta\beta\sqrt{\frac{2\pi}{d}} + t\zeta\sqrt{\frac{2\pi}{d}} - t\beta\frac{2\pi}{d} - \frac{\pi t^2}{d}]$$
$$\sum_{m=0}^{d-1} \widetilde{\mathcal{G}}_m \sum_{L_n=-\infty}^{\infty} \exp(\frac{2i\pi m L_n}{d}) \exp(2L_n\zeta_n\sqrt{\frac{\pi}{2d}} - 2L_n\beta\frac{\pi}{d} - \frac{\pi L_n^2}{d} - 2\frac{\pi L_n t}{d}) \tag{4.63}$$

$$G(\zeta_n - \beta\sqrt{\frac{2\pi}{d}}; t; \mathcal{G}) = \pi^{\frac{-1}{4}} \exp(-\frac{\zeta_n^2}{2}) \exp(-\frac{\pi}{d}(\beta^2 + 2t\beta + t^2)) \exp(\zeta\sqrt{\frac{2\pi}{d}}(\beta + t))$$
$$\sum_{m=0}^{d-1} \widetilde{\mathcal{G}}_m \sum_{L_n=-\infty}^{\infty} \exp(\frac{2i\pi m L_n}{d}) \exp(2L_n\zeta_n\sqrt{\frac{\pi}{2d}} - \frac{L_n^2\pi}{d} - \frac{2\pi L_n}{d}(\beta + t)) \tag{4.64}$$

Let $\beta + t = \Gamma$

$$G(\zeta_n - \beta\sqrt{\frac{2\pi}{d}}; t; \mathcal{G}) = \pi^{\frac{-1}{4}} \exp(-\frac{\zeta_n^2}{2}) \exp(-\frac{\pi}{d}\Gamma^2) \exp(\zeta\sqrt{\frac{2\pi}{d}}\Gamma)$$
$$\sum_{m=0}^{d-1} \widetilde{\mathcal{G}}_m \sum_{L_n=-\infty}^{\infty} \exp(\frac{2i\pi m L_n}{d}) \exp(2L_n\zeta_n\sqrt{\frac{\pi}{2d}} - \frac{L_n^2\pi}{d} - \frac{2\pi L_n}{d}\Gamma). \tag{4.65}$$

We insert $z = \zeta_{n+\beta}(t + \beta)$ in $G(z;t)$, and using $G[\zeta_n(t);t] = 0$ we prove that $G[\zeta_{n+\beta}(t + \beta); t + \beta] = 0$.

In this part we plot the paths of the zeros of the state $\mathfrak{X}^t|\mathcal{G}\rangle$ shown in Fig (4.13). The state $|\mathcal{G}\rangle$ is defined through the zeros at $t = 0$ which are:

$$\zeta_0(0) = 2.1708 + 0.7208i;\ \zeta_1(0) = 2.1708 + 2.1708i;\ \zeta_2(0) = 2.1708 + 3.6208i \tag{4.66}$$

We can see that there are d identical paths, which are shifted in both direction (in the real and imaginary axis) z_R and z_I-direction by $(\sqrt{2\pi/d})$.

In Fig (4.14), we plot the paths of the zeros of $G(z, t)$ which represents the state $[\mathcal{D}(2, 1)]^t|\mathcal{G}\rangle$. The state $|\mathcal{G}\rangle$ is defined through the zeros at $t = 0$, which are

$$\zeta_0(0) = 1.46 + 2.44i;\ \zeta_1(0) = 2.1 + 2.21i;\ \zeta_2(0) = 2.95 + 1.85i. \tag{4.67}$$

In this case we seen that one of the paths has multiplicity $M = 2$ and the other one has multiplicity $M = 1$

$$\begin{aligned} \zeta_0(T) = \zeta_1(0); \quad \zeta_1(T) = \zeta_0(0); \\ \zeta_2(T) = \zeta_2(0). \end{aligned} \tag{4.68}$$

So we can note that at a particulate time the zeros do not obey the relation $\zeta_{i+1}(t) = \zeta_i(t) + \sqrt{2\pi/d}$ (e.g., the zeros at $t = 0$ which are shown in Fig (4.13) and in Fig (4.14). In this statue the whole path of a zero over a period T, is a shifted of the path of another zero.

The displacement operators in the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space, are defined as $\mathcal{D}(\alpha, \beta) = \mathfrak{Z}^{\alpha}\mathfrak{X}^{\beta}\omega(-2^{-1/2})$. So we assume that d is an odd integer, where 2^{-1} exists in $\mathbb{Z}(d)$.
Let e_m and $|u_m\rangle$ be the eigenvalues and eigenvectors of $\mathcal{D}(\alpha, \beta)$. We consider the state $|\mathcal{G}\rangle = \sum r_m|u_m\rangle$. Then we assume that the state $[\mathcal{D}(\alpha, \beta)]^t|\mathcal{G}\rangle = \sum e_m^t|u_m\rangle$ is represented by the function $G(z, t)$.
Let $\zeta_n(t)$ where $(n = 0, ..., d-1)$ be the zeros of $G(z; t)$, i.e.

$$G[\zeta_n(t); t] = 0 \tag{4.69}$$

In this section we speculate that, each path of the zeros of $G(z, t)$, is a shifted version (in both directions, real and imaginary directions) of another path. This assumption is supported by the numerical result in Fig(4.15), where the paths of the zeros of the function $G(z, t)$ are plotted which represents the state $[\mathcal{D}(1, 1)]^t|\mathcal{G}\rangle$. We also define the state $|\mathcal{G}\rangle$ through the zeros at $t = 0$ as following:

$$\zeta_0(0) = 1.01 + 2i;\ \zeta_1(0) = 2.15 + 2.56i;\ \zeta_2(0) = 3.35 + 1.95i. \tag{4.70}$$

We can see that there are d identical paths, which are shifted in the both direction by $\sqrt{2\pi/d}$.

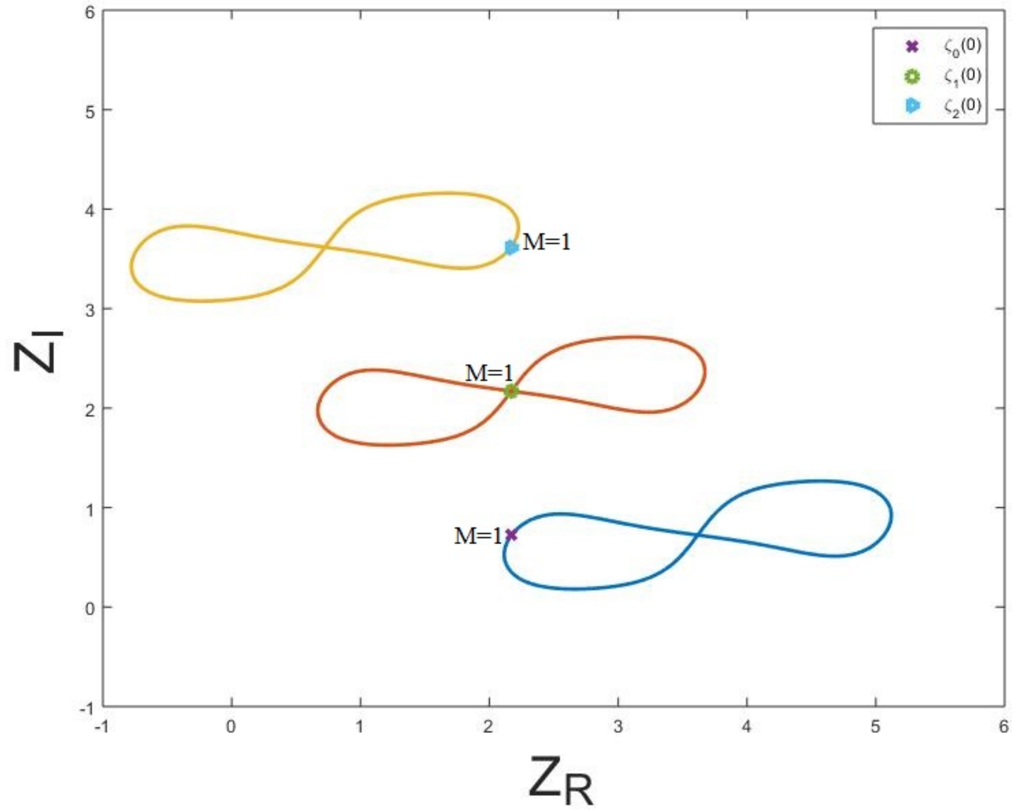

Figure 4.13: Paths of the zeros of the analytic representation of the state $[\mathcal{D}(1,1)]^t|\mathcal{G}\rangle$. The state $|\mathcal{G}\rangle$ is defined through the zeros at $t = 0$ given in Eq.(4.66) The cell is a square with each side equal to 4.3.

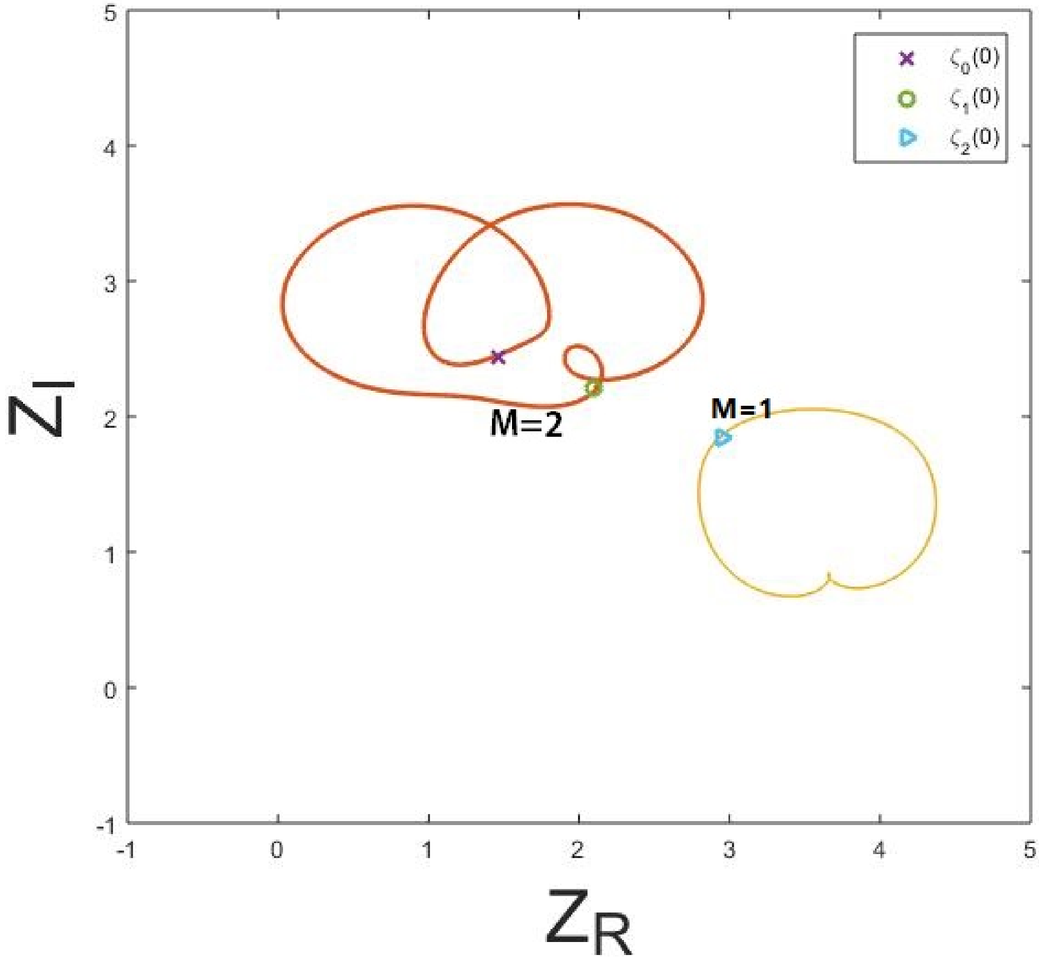

Figure 4.14: Paths of the zeros of the analytic representation of the state $[\mathcal{D}(2,1)]^t|\mathcal{G}\rangle$. The state $|\mathcal{G}\rangle$ is defined through the zeros at $t = 0$ given in Eq.(4.67) The cell is a square with each side equal to 4.3.

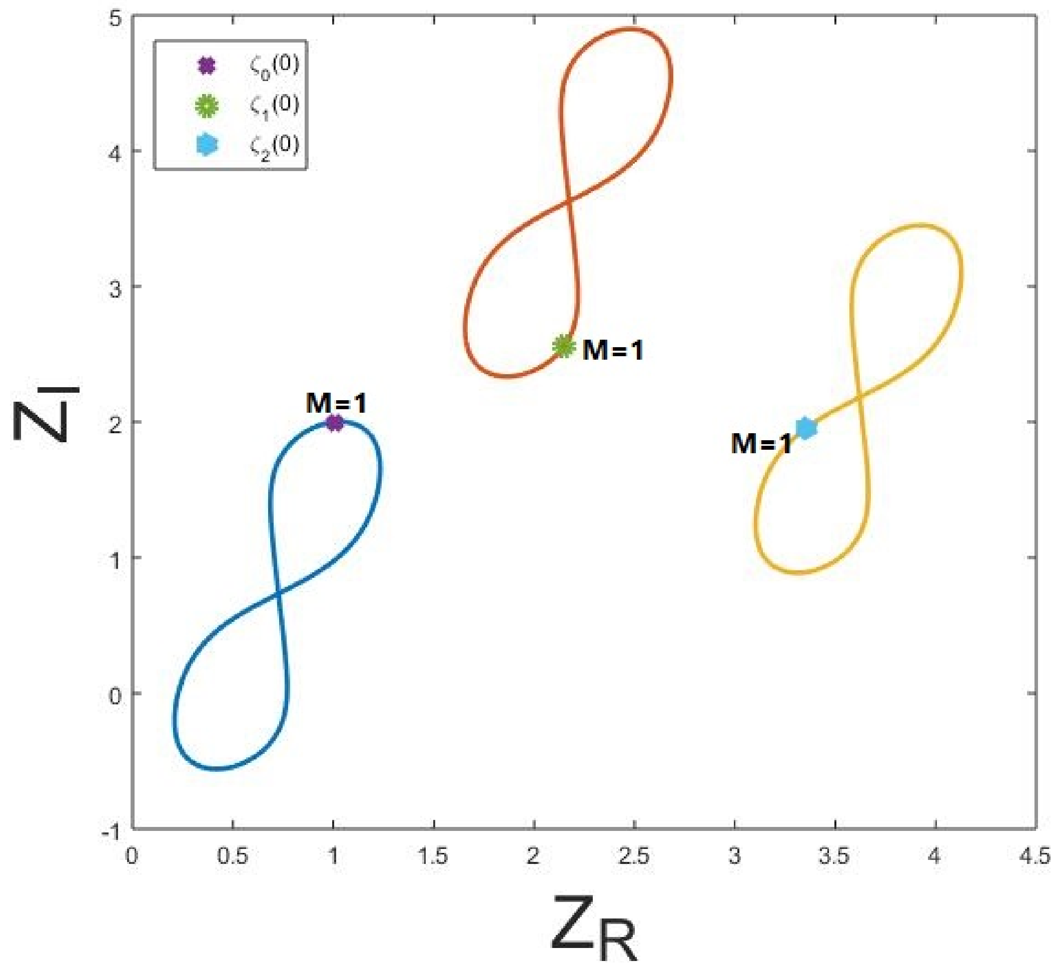

Figure 4.15: Paths of the zeros of the analytic representation of the state $[\mathcal{D}(1,1)]^t|\mathcal{G}\rangle$. The state $|\mathcal{G}\rangle$ is defined through the zeros at $t = 0$ given in Eq.(4.70) The cell is a square with each side equal to 4.3.

4.4 Conclusion

We investigated quantum systems with finite Hilbert space for odd values of d, where positions and momentum take values in $\mathbb{Z}(d)$. Quantum states were represented with the analytic function on a torus that uses Theta functions, which describes these systems, given in Eq(4.10). As the system evolves in time the zeros follow d paths on the torus. A method for the calculation of these paths of zeros has been given in Eq(4.22), and (4.23). Each path is characterized by the multiplicity M, and by a pair of winding numbers (w_1, w_2). We also studied other phenomena such as the joining of two paths of zeros into a single path. Furthermore, we considered the periodic system which has the displacement operator to real power t, as time evolution operator. Several numerical examples, that illustrates these ideas were provided in order to explore the behaviour.

Chapter 5

Conclusion and Future work

There are deep links between the behaviour of a quantum system and the zeros of the analytic representation of these systems.

We considered phase space methods for quantum systems with d dimensional Hilbert space. An analytic representation on a torus using Theta functions was studied. In finite quantum systems, the d zeros of these analytic functions define uniquely the quantum state of the system. These zeros obey the constraint of Equation(4.1). The zeros are used to describe the time evolution of these system. For the time dependent system, the d zeros follow d paths on the torus. As the system evolves in time, the d zeros following d paths on the torus, which define the Hamiltonian.

The calculation of the paths of the zeros using a analytic method was given in Equations(4.22), and (4.23), and has been used to study the evolution of periodic systems. The work shows that the paths of the zeros provide a substitute description for the standard quantum formalism. We have shown that each path is characterized by a pair of winding numbers (w_1, w_2). We also considered the case of paths with multiplicity M, and presented examples with $M = 1; 2; 3; .....$ It was established that the zeros after a period the zeros exchange their positions (Eq(4.28),(4.38),(4.43),(4.46)). Other important phenomena such as the joining of two paths of zeros into a single path were considered. We gave various examples of the paths $\zeta_n(t)$ of the zeros, for various Hamiltonians. The situation where the time evolution operator is the displacement operator to the real power t, has also been studied. In this case each paths of zeros of the function $G(z, t)$ in Equation (4.55) are shifted version of another path. Several numerical example that illustrates these idea were provided in order to explore the behaviour.

We also studied the Bargmann analytic representations of system on $\mathbb{R}$. As an example of a Bargmann function we used the Mittag-Leffler function $E_{\alpha,\beta}(z)$ for the arbitrary complex argument $z \in \mathbb{C}$ and parameters $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. We study the growth of Bargmann function using Mittag-Leffler function as a good example. We calculate the number of the zeros using numerical method and have found that the Mittag-Leffler function might have finite number of zeros when $1 < \alpha < 2$. The quantum statistical properties of the corresponding quantum states have been studied.

The work can be extended to study the behaviour of more complex periodic systems through the paths of the zeros of analytic representation. The work may be also extended to consider the analytic representation using infinite quantum system, to study the effect of a change in the basis on the paths of the zeros .

Bibliography

[1] V. Bargmann. On a Hilbert space of analytic functions and an associated integral transform Part I. Commun. Pure Appl. Math. **14**, 187 (1961).

[2] P. Leboeuf, A. Voros, J. Phys. A23, 1765 (1990)

[3] P. Leboeuf, J. Phys. A24, 4575 (1991)

[4] M.B. Cibils, Y. Cuche, P. Leboeuf, W.F. Wreszinski, Phys. Rev A46, 4560 (1992)

[5] S. Nonnenmacher, A. Voros, J. Phys. A30, L677 (1997)

[6] H.J. Korsch, C. Múller, H. Wiescher, J. Phys. A30, L677 (1997)

[7] F. Toscano, A.M.O. de Almeida, J. Phys. A32, 6321 (1999)

[8] D. Biswas, S. Sinha, Phys. Rev. E60, 408 (1999)

[9] A. Perelomov, 'Generalized coherent states and their applications', (Springer, Berlin, 1986)

[10] B.C. Hall, Contemp. Math. 260, 1 (2000).

[11] R. J. Glauber. The quantum theory of optical coherence. Phys. Rev.,130:2529, (1963).

[12] R. J. Glauber. Coherent and incoherent states of the radiation eld.Phys. Rev., 131:2766, (1963).

[13] E. C. G. Sudarshan. Equivalence of semiclassical and quantum mechanical descriptions of statistical ligth beams. Phys. Rev. lett., 10:277, (1963).

[14] D. F. Walls and G. Milburn. Quantum Optics. Springer, Berlin, (1994).

[15] J. R. Klauder and E. C. G. Sudarshan. Coherent States.World Scientic,Singapore, (1985).

[16] W. Vogel and D. G. Welsch. Lectures on Quantum Optics. Akademie-Verlag, Berlin, (1990).

[17] P. Woit.Quantum Theory, Groups and Representations:An Introduction (under construction).Columbia University.March 13, (2016).

[18] A. Vourdas, J. Phys. A39, R65 (2006)

[19] A. Vourdas, The growth Bargmann functions and the completeness of sequences of coherent states. J. Phys. A: Math. Gen., 30:4867, (1997).

[20] A. Vourdas, K.A.Penson,G.H.Duchamp and A.I.Solomon ,Generalized Bargmann functions, their growth and von Neumann lattices. J. Phys. A: Math, 45.244031-244039,(2012).

[21] A. Vourdas, Phys. Rev. A, 41:1653 (1990).

[22] A. Vourdas, Phys. Rev. A, 43:1564 (1991).

[23] A. Vourdas and C. Bendjaballah, Phys. Rev. A, 47:3523 (1993).

[24] A. Vourdas, J. Phys. A, 36:5645 (2003).

[25] V. Bargmann. On a Hilbert space of analytic functions and an associated integral transform Part I. Commun. Pure Appl. Math. 14:187,(1961).

[26] V. Bargmann. On a Hilbert space of analytic functions and an associated integral transform Part II. Commun. Pure Appl. Math., 120:1, (1967).

[27] H. Weyl, Theory of Groups and Quantum Mechanics, New York: Dover, (1950).

[28] E. Wigner. On the quantum correction for thermodynamic equilibrium.Phys. Rev., 40:749, (1932).

[29] H. Groenewold. On the principles of elementary quantum mechanics.Physica, 12:405, (1946).

[30] Y. S. Kim and M. E. Noz. Phase Space Picture of Quantum Mechanics.Worlf Scientic, (1991).

[31] C. Zachos. Deformation quantization: Quantum mechanics lives and works in phase-space. Int. J. Mod. Phys. A, 17:297, (2002).

[32] K.B. Wolf. Wigner distribution function for paraxial polychromatic optics.Optics Communications 132, 343-352(1996).

[33] Hu Li-Yun and Fan Hong-Yi.Wigner function of the thermo number states.Chinese Physics .18, 3, March (2009).

[34] J. Schwinger, Quantum Kinematics and Dynamics, New York: Benjamin, (1970).

[35] W. K. Wooters, Ann. Phys. (NY), 176:1 (1987).

[36] W. K. Wooters and B. D. Fields, Ann. Phys. (NY), 191:363 (1989).

[37] M. L. Mehta, J. Math Phys., 28:781 (1987).

[38] D. Galetti and A. F. R. de Toledo-Piza, Physica A, 149:267 (1988).

[39] J. C. Varilly and J. M. Gracia-Bondia, Ann. Phys. (NY), 190:107 (1989).

[40] H. Figueroa, J. M. Gracia-Bondia, and J. C. Varilly, J. Math. Phys., 31:2664 (1990).

[41] T. Lulek, Acta Phys. Polon. A, 82:377 (1992).

[42] T. Lulek, Rep. Math. Phys., 34:71 (1994).

[43] U. Leonhardt, Phys. Rev. Lett., 74:4101 (1995).

[44] U. Leonhardt, Phys. Rev. A, 53:2998 (1996).

[45] J. Schwinger, 'Quantum Kinematics and Dynamics' (New York, Benjamin), (1970).

[46] P. Stovicek, J. Tolar, Rep. Math. Phys. 20, 157 (1984)

[47] A. Vourdas, Rep. Prog. Phys., 67:267 (2004)

[48] M. Kibler, J. Phys. A42, 353001 (2009).

[49] L. Auslander and R. Tolimieri, Bull. Am. Math. Soc., 1:847 (1979)

[50] A. Vourdas and C. Banderier, J. Phys. A: Math. Theor., 43:042001(2010)

[51] S. Zhang, A. Vourdas, J. Phys. A37, 8349 (2004); and corrigendum in J. Phys. A38, 1197 (2005)

[52] M. Tubani, A. Vourdas, S. Zhang, Phys. Scr. 82, 038107 (2010).

[53] R. P. Boas. Entire Functions. Academic Press, New York, (1954).

[54] P. Feinsilver, Monatshefte fr Mathematik, 104:89 (1987)

[55] L.D. Faddeev, Lett. in Math. Phys., 34:249 (1995).

[56] A. Ballesteros, F.J. Herranz and P. Parashar, J. Phys. A: Math. Gen, 30:L149 (1997)

[57] P. Evangelides, C. Lei, A. Vourdas, J. Math.Phys. 56, 072108 , 1-16(2015).

[58] P. Evangelides. Analytic representation of quantum system with theta functions,Bradford university. (2015).

[59] I. Segal. Mathematical characterization of the physical vacuum for a linear Bose-Einstein eld. Illinois. J. Math., 6:500, (1962).

[60] A. Tyurin, 'Quantization, classical and quantum field theory and theta functions' ,(American Math. Society, Rhode Island), (2003).

[61] H. Eissa, P. Evangelides, C. Lei, A. Vourdas, Physics Letters A380, 548553(2016).

[62] M. Ruzzi, J. Math. Phys. 47, 063507 (2006).

[63] R Bellman, 'A Brief Introduction to Theta Functions', Dover Publication, Inc.Mineola,New York,(2013).

[64] D Mumford, 'Tata lectures on Theta', Vols 1,2,3 (Birkhauser, Boston, 1983)

[65] A.A.Kilbas, M Saigo,R.K. Saxena. 'Generalized Mittag-Leffler function and generalized fractional calculus operators'. Integral Transforms Spec. Funct. 15(1), 3149 (2004).

[66] R.K.Saxena, A.M.Mathai, H.J.Haubold. On generalized fractional kinetic equations. Phys. A, Stat. Mech. Appl. 344, 657664 (2004).

[67] A. Erdelyi(Ed.), Higher Transcendental Functions,Bateman project, vol. 1-3,McGraw-Hill, New York, NY, USA, (1955).

[68] K.S. Cole, Electrical conductance of biological systems, in: Proceedings of the Symposium on Quantitative Biology,Cold Spring Harbor, New York, 107-116,(1933).

[69] B. Gross, On creep and relaxation, J. Appl. Phys. 18 , 212-221(1947).

[70] M. Caputo, F. Mainardi, Linear models of dissipation in anelastic solids, Riv. Nuovo Cimento (Ser. II) 1, 161-198 (1971).

[71] R.Gorenflo,A. Kilbas, F. Mainardi, S. V. Rogosin, 'Mittag-Leffler Functions, Related Topics and Applications',(Springer, Berlin,2014).

[72] I.Grattan-Guinness, 'Materials for the History of Mathematics in the Institute Mittag-Leffler ',Isis,62,3,363-374(Autumn,1971).

[73] F. Mainardi,On some properties of the Mittag-Leffler function $E_\alpha(-t^\alpha)$, completely monotone for $t > 0$ with $0 < \alpha < 1$,Mathematical Physics (math-ph); arXiv:1305.0161 [math-ph].

[74] A. Wiman, 'ber den Fundamentalsatz der Theorie der Funkntionen $E_\alpha(x)$. Acta Math. 29, 191201 (1905)

[75] A.A.Kilbas, M.Saigo and R.K.Saxena , Solution of volterra integro differential Equations with generalized Mittag-Leffler function in the kernels, Journal OF Integral Equations and applications,14, 4, (2002).

[76] J.W.Hanneken, B.N.Achar, R.Puzio , and D.M.Vaught,'Properties of the MittagLeffler function for negative alpha', Physica Scripta, (2009).

[77] R. Gorenflo, Y. Luhko,and S. Rogosin,Mittag-Leffler Type Functions: Notes on Growth Properties and Distribution of Zeros, Freie Univ, Fachbereich Mathematik und Informatik, (1997).

[78] F.Mainardi, Yu. Luchko,G. Pagnini . 'The fundamental solution of the spacetime fractional diffusion equation'. Fract. Calc. Appl. Anal. 4, 153192 (2001).

[79] T.R.Prabhakar, 'A singular integral equation with a generalized Mittag-Leffler function in the kernel'. Yokohama Math. J. 19, 715 (1971)

[80] S. Rogosin , A. Koroleva, 'Integral representation of the four-parametric generalized Mittag-Leffler function'. Lithuanian Mathematical Journal, 50, 3,337343(2010).

[81] B. J. Levin, 'Distribution of Zeros of Entire Functions', Amer. Math. Soc.,Providence, (1980).

[82] B. J. Levin, Lecture on Entire Functions. Rhode Island: American Mathematical Society, (1996).

[83] M. Marden, Geometry of Polynomials, Second edition. Mathematical Surveys, No. 3, American Mathematical Society, Providence, R.I., (1966).

[84] A. R. Reddy , 'On the Distribution of Zeros of Entire Functions', Proceedings of the American Mathematical Society,45, 1 ,105-112(Jul,1974).

[85] P.Raymond, H.John, A.Narahari, 'Zeros of Mittag-Leffler function' , American Physical Society, (2004).

[86] R.Sergei.'The Role of the Mittag-Leffler Function in Fractional Modeling', Mathematics, 3, 368-381 (2015).

[87] W.Hanneken, M.Vaught,and.B.N.Achar. 'Enumeration of the real zeros of the mittag-leffler function $E_\alpha(z)$, $1 < \alpha < 2$', J. Sabatier et al(eds),Advances in Fractional Calculus. 'Theoretical Developments and Applications in Physics and Engineering',15-26,(2007).

[88] J.Sheng, Z.Wang, S.Zhong Fu, The zeros of the solutions of the fractional oscillation equation, Journal for theory and Applications, 17,(2014).

[89] A.R.Vargas, Zeros of sections of power series, arXiv:1208.5186v2 [math.NT] (2012).